

# NON-UNIPOTENT REPRESENTATIONS AND CATEGORICAL CENTRES

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## INTRODUCTION

**0.1.** Let  $\mathbf{k}$  be an algebraic closure of the finite field with  $p$  elements. Let  $G$  be a connected reductive group over  $\mathbf{k}$ . We denote by  $F_q$  the subfield of  $\mathbf{k}$  with exactly  $q$  elements; here  $q$  is a power of  $p$ . Let  $F : G \rightarrow G$  be the Frobenius map for an  $F_q$ -rational structure on  $G$ . We fix a prime number  $l$  different from  $p$ . Let  $\text{Irr}(G^F)$  be the set of isomorphism classes of irreducible representations (over  $\bar{\mathbf{Q}}_l$ ) of the finite group  $G^F = \{g \in G; F(g) = g\} = G(F_q)$ . In [L2] I gave a parametrization of  $\text{Irr}(G^F)$  in terms of the group of type dual to that of  $G$ . (For “most” representations in  $\text{Irr}(G^F)$  this has been already done in [DL].) For the part of  $\text{Irr}(G^F)$  consisting of unipotent representations in a fixed two-sided cell of  $W$  (with  $G$  assumed to be  $F_q$ -split) the parametrization was in terms of a set  $M(\Gamma)$  where  $\Gamma$  is a certain finite group associated to the two-sided cell and  $M(\Gamma)$  is the set of simple objects (up to isomorphism) of the category  $\text{Vec}_\Gamma(\Gamma)$  of  $\Gamma$ -equivariant vector bundles on  $\Gamma$  (here  $\Gamma$  acts on  $\Gamma$  by conjugation). In the early 1990’s, Drinfeld pointed out to me that the category  $\text{Vec}_\Gamma(\Gamma)$  can be interpreted as the categorical centre of the monoidal category of finite dimensional representations of  $\Gamma$ . (The notion of categorical centre of a monoidal category is due to Joyal, Street, Majid and Drinfeld.) This suggested that one should be able to reformulate the parametrization of  $\text{Irr}(G^F)$  in terms of categorical centres of suitable monoidal categories associated with  $G$ . This is achieved in the present paper, except that we must allow certain twisted categorical centres instead of usual categorical centres. Note that in our approach the representation theory of  $G(F_q)$  cannot be separated from the theory of character sheaves on  $G$  which appears as the limit of the first theory when  $q$  tends to 1; in particular we also obtain the parametrization of character sheaves on  $G$  in terms of categorical centres (no twisting needed in this case).

Earlier results of this type were known in the following cases:

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Supported by NSF grant DMS-1566618.

- (i) the case [BBD] of character sheaves on  $G$  (with centre assumed to be connected and with  $\mathbf{k}$  replaced by  $\mathbf{C}$ );
- (ii) the case [L14] of unipotent character sheaves on  $G$ ;
- (iii) the case [L15] of unipotent representations of  $G^F$ ;
- (iv) the case [L16] of not necessarily unipotent character sheaves on  $G$ .

The papers [L15],[L16] were generalizations of [L14] in different directions; the present paper is a common generalization of [L15],[L16]; the methods used in (ii),(iii),(iv) and the present paper are quite different from those used in (i) which relied on techniques not available in positive characteristic.

Let  $\mathbf{B}$  be a Borel subgroup of  $G$  and let  $\mathbf{T}$  be a maximal torus of  $\mathbf{B}$ . In this subsection we assume that  $F(\mathbf{B}) = \mathbf{B}$ ,  $F(\mathbf{T}) = \mathbf{T}$ . Let  $W$  be the Weyl group of  $G$  with respect to  $\mathbf{T}$ . Let  $\mathfrak{s}$  be an indexing set for the isomorphism classes of Kummer local systems (over  $\bar{\mathbf{Q}}_l$ ); note that  $W$  acts naturally on  $\mathfrak{s}$ .

Let  $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$  where  $v$  is an indeterminate. A key role in this paper is played by an  $\mathcal{A}$ -algebra  $\mathbf{H}$  (without 1 in general) which has  $\mathcal{A}$ -basis  $\{T_w 1_\lambda; w \in W, \lambda \in \mathfrak{s}\}$  and multiplication defined in 1.5 (see also [L9, 31.2]). This is a monodromic version of the usual Hecke algebra of  $W$ , closely related to an algebra defined in [Yo]; it contains the usual Hecke algebra as a subalgebra. Now  $\mathbf{H}$  has a canonical basis, two-sided cells and an asymptotic version  $H^\infty$  (introduced in [L10],[L16]) which generalize the analogous notions for the usual Hecke algebra, see [KL], [L3]; the two-sided cells form a partition of  $W \times \mathfrak{s}$  and we have  $H^\infty = \bigoplus_{\mathbf{c}} H_{\mathbf{c}}^\infty$  as rings ( $\mathbf{c}$  runs over the two-sided cells and each  $H_{\mathbf{c}}^\infty$  is a ring with 1). For any  $\mathbf{c}$ ,  $H_{\mathbf{c}}^\infty$  admits a category version (for which  $H^\infty$  is the Grothendieck group) which is a semisimple monoidal category  $\mathcal{C}^{\mathbf{c}}$  with finitely many simple objects (up to isomorphism) indexed by the elements of  $\mathbf{c}$ , see §5. (In the case where  $\mathbf{c} \subset W \times \{1\}$ , this reduces to the monoidal category defined in [L7].) Now  $\mathcal{C}^{\mathbf{c}}$  has a well defined categorical centre which is again a semisimple abelian category. Note that  $F$  acts naturally on  $\mathfrak{s}$  and on  $W$  hence on  $W \times \mathfrak{s}$ ; this induces an action of  $F$  on the set of two-sided cells. If  $\mathbf{c}$  is a two-sided cell such that  $F(\mathbf{c}) = \mathbf{c}$  then  $F$  defines an equivalence of categories  $\mathcal{C}^{\mathbf{c}} \rightarrow \mathcal{C}^{\mathbf{c}}$  and one can define the notion of  $F$ -centre of  $\mathcal{C}^{\mathbf{c}}$  (see 5.5) which is a twisted version of the usual centre; it is a semisimple abelian category. We denote by  $[\mathbf{c}]$  the set of isomorphism classes of simple objects of this category (a finite set).

Our main result is that  $\text{Irr}(G^F)$  is in natural bijection with  $\sqcup_{\mathbf{c}} [\mathbf{c}]$  (disjoint union over all  $F$ -stable two-sided cells  $\mathbf{c}$ ). (See Theorem 7.3.) In the case where  $\mathbf{c} \subset W \times \{1\}$ , this reduces to the main result in [L15].

The fact that the asymptotic Hecke algebra  $\mathbf{H}^\infty$  plays a role in the classification is perhaps not surprising since its non-monodromic versions appeared implicitly in the arguments of [L1], through the traces of their canonical basis elements in their various simple modules (the algebras themselves were not defined at the time where [L1] was written).

Many arguments in this paper follow very closely the arguments in [L16]; we generalize them by taking into account also the arguments in [L15]. We have

written the proofs in such a way that they apply at the same time in the case of character sheaves on a connected component of a possibly disconnected algebraic group with identity component  $G$ . In this case, the classification involves twisted categorical centers, unlike that for the character sheaves on  $G$ .

We plan to show elsewhere that the parametrization of  $\text{Irr}(G^F)$  given in [L2] can be deduced from the main result of this paper.

**0.2. Notation.** Let  $\mathbf{N}^* = \{n \in \mathbf{Z} - p\mathbf{Z}; n \geq 1\}$ . Let  $T$  be a torus over  $\mathbf{k}$ . For  $n \in \mathbf{N}^*$  let  $T_n = \{t \in T; t^n = 1\}$ ; we have  $\sharp(T_n) = n^{\dim T}$ . For  $n, n'$  in  $\mathbf{N}^*$  such that  $n'/n \in \mathbf{Z}$  we have a surjective homomorphism  $N_n^{n'} : T_{n'} \rightarrow T_n, t \mapsto t^{n'/n}$ . Hence we can form the projective limit  $T^\infty$  of the groups  $T_n$  with  $n \in \mathbf{N}^*$  (a profinite abelian group). Then for any  $n \in \mathbf{N}^*$ ,  $T_n$  is naturally a quotient of  $T^\infty$ .

All algebraic varieties are over  $\mathbf{k}$ . We denote by  $\mathbf{p}$  the algebraic variety consisting of a single point. For an algebraic variety  $X$  we write  $\mathcal{D}(X)$  for the bounded derived category of constructible  $\bar{\mathbf{Q}}_l$ -sheaves on  $X$ . Let  $\mathcal{M}(X)$  be the subcategory of  $\mathcal{D}(X)$  consisting of perverse sheaves on  $X$ . For  $K \in \mathcal{D}(X)$  and  $i \in \mathbf{Z}$  let  $\mathcal{H}^i K$  be the  $i$ -th cohomology sheaf of  $K$  and let  $K^i$  be the  $i$ -th perverse cohomology sheaf of  $K$ . Let  $\mathfrak{D}(K)$  be the Verdier dual of  $K$ . For any constructible sheaf  $\mathcal{E}$  on  $X$  let  $\mathcal{E}_x$  be the stalk of  $\mathcal{E}$  at  $x \in X$ . If  $X$  has a fixed  $F_q$ -structure  $X_0$ , we denote by  $\mathcal{D}_m(X)$  what in [BBD, 5.1.5] is denoted by  $\mathcal{D}_m^b(X_0, \bar{\mathbf{Q}}_l)$ ; let  $\mathcal{M}_m(X)$  be the corresponding category of mixed perverse sheaves. In this paper we often encounter maps of algebraic varieties which are not morphisms but only quasi-morphisms (as in [L15, 0.3]). For such maps the usual operations with derived categories are defined as in [L15, 0.3].

Note that if  $K \in \mathcal{D}_m(X)$  then  $K$  can be viewed as an object of  $\mathcal{D}(X)$  denoted again by  $K$ . If  $K \in \mathcal{M}_m(X)$  and  $h \in \mathbf{Z}$ , we denote by  $gr_h(K)$  the subquotient of pure weight  $h$  of the weight filtration of  $K$ . If  $K \in \mathcal{D}_m(X)$  and  $i \in \mathbf{Z}$  we write  $K\langle i \rangle = K[i](i/2)$  where  $[i]$  is a shift and  $(i/2)$  is a Tate twist; we write  $K\{i\} = gr_i(K^i)(i/2)$ . If  $K$  is a perverse sheaf on  $X$  and  $A$  is a simple perverse sheaf on  $X$  we write  $(A : K)$  for the multiplicity of  $A$  in a Jordan-Hölder series of  $K$ . If  $C \in \mathcal{D}_m(X)$  and  $\{C_i; i \in I\}$  is a family of objects of  $\mathcal{D}_m(X)$  then the relation  $C \simeq \{C_i; i \in I\}$  is as in [L16, 0.2].

Let  $\bar{\cdot} : \mathcal{A} \rightarrow \mathcal{A}$  be the ring homomorphism such that  $\overline{v^m} = v^{-m}$  for any  $m \in \mathbf{Z}$ . If  $f \in \mathbf{Q}[v, v^{-1}]$  and  $j \in \mathbf{Z}$  we write  $(j; f)$  for the coefficient of  $v^j$  in  $f$ .

Let  $\mathcal{B}$  be the variety of Borel subgroups of  $G$ . For any  $B \in \mathcal{B}$  let  $U_B$  be the unipotent radical of  $B$ . In this paper we fix a Borel subgroup  $\mathbf{B}$  of  $G$  and a maximal torus  $\mathbf{T}$  of  $\mathbf{B}$ . Let  $\mathbf{U} = U_{\mathbf{B}}$ . Let  $\nu = \dim \mathbf{U} = \dim \mathcal{B}$ ,  $\rho = \dim \mathbf{T}$ ,  $\Delta = \dim G = 2\nu + \rho$ .

For any algebraic variety  $X$  let  $\mathfrak{L} = \mathfrak{L}_X = \alpha_! \bar{\mathbf{Q}}_l \in \mathcal{D}(X)$  where  $\alpha : X \times \mathbf{T} \rightarrow X$  is the obvious projection. When  $X$  and  $T$  are defined over  $\mathbf{F}_q$ ,  $\mathfrak{L}$  is naturally an object of  $\mathcal{D}_m(X)$ .

Unless otherwise specified, all vector spaces are over  $\bar{\mathbf{Q}}_l$ ; in particular, all representations of finite groups are assumed to be in (finite dimensional)  $\bar{\mathbf{Q}}_l$ -vector

spaces.

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### 1. THE MONODROMIC HECKE ALGEBRA AND ITS ASYMPTOTIC VERSION

**1.1.** Let  $N\mathbf{T}$  be the normalizer of  $\mathbf{T}$  in  $G$ , let  $W = N\mathbf{T}/\mathbf{T}$  be the Weyl group and let  $\kappa : N\mathbf{T} \rightarrow W$  be the obvious homomorphism. For  $w \in W$  we set  $G_w = \mathbf{U}\kappa^{-1}(w)\mathbf{U}$  so that  $G = \sqcup_{w \in W} G_w$ ; let  $\mathcal{O}_w = \{(x\mathbf{B}x^{-1}, y\mathbf{B}y^{-1}); x \in G, y \in G, x^{-1}y \in G_w\}$  so that  $\mathcal{B} \times \mathcal{B} = \sqcup_{w \in W} \mathcal{O}_w$ . For  $w \in W$  let  $\bar{G}_w$  be the closure of  $G_w$  in  $G$ ; we have  $\bar{G}_w = \cup_{y \leq w} G_y$  for a well defined partial order  $\leq$  on  $W$ . Let  $\bar{\mathcal{O}}_w$  be the closure of  $\mathcal{O}_w$  in  $\mathcal{B} \times \mathcal{B}$ . Now  $W$  is a (finite) Coxeter group with length function  $w \mapsto |w| = \dim \mathcal{O}_w - \nu$  and with set of generators  $S = \{\sigma \in W; |\sigma| = 1\}$ ; it acts on  $\mathbf{T}$  by  $w : t \mapsto w(t) = \omega t \omega^{-1}$  where  $\omega \in \kappa^{-1}(w)$ .

**1.2.** Let  $R \subset \text{Hom}(\mathbf{T}, \mathbf{k}^*)$  be the set of roots of  $G$  with respect to  $\mathbf{T}$ . Now  $W$  acts on  $R$  by  $w : \alpha \mapsto w(\alpha)$  where  $(w(\alpha))(t) = \alpha(w^{-1}(t))$  for  $t \in \mathbf{T}$ . Let  $R^+$  be the set of  $\alpha \in R$  such that the corresponding root subgroup is contained in  $G$ . For  $\alpha : \mathbf{T} \rightarrow \mathbf{k}^*$  we denote by  $\check{\alpha} : \mathbf{k}^* \rightarrow \mathbf{T}$  the corresponding coroot and by  $\sigma_\alpha$  the corresponding reflection in  $W$ . For any  $\sigma \in S$  let  $\mathbf{U}_\sigma$  be the unique root subgroup of  $\mathbf{U}$  with respect to  $\mathbf{T}$  such that  $\mathbf{U}_\sigma^- := \omega \mathbf{U}_\sigma \omega^{-1} \not\subset \mathbf{U}$  for some/any  $\omega \in \kappa^{-1}(\sigma)$ . Let  $\alpha_\sigma : \mathbf{T} \rightarrow \mathbf{k}^*$  be the root corresponding to  $\mathbf{U}_\sigma$ ; then the coroot  $\check{\alpha}_\sigma : \mathbf{k}^* \rightarrow \mathbf{T}$  is well defined.

For any  $\sigma \in S$  we fix an element  $\xi_\sigma \in \mathbf{U}_\sigma - \{1\}$ ; there is a unique  $\xi'_\sigma \in \mathbf{U}_\sigma^- - \{1\}$  such that  $\xi_\sigma \xi'_\sigma \xi_\sigma = \xi'_\sigma \xi_\sigma \xi'_\sigma \in \kappa^{-1}(\sigma) \subset N\mathbf{T}$ ; the two sides of the last equality are denoted by  $\dot{\sigma}$ . We have  $\kappa(\dot{\sigma}) = \sigma$  and  $\dot{\sigma}^2 = \check{\alpha}_\sigma(-1)$ . For any  $w \in W$  we define  $\dot{w} \in N\mathbf{T}$  by  $\dot{w} = \dot{\sigma}_1 \dot{\sigma}_2 \dots \dot{\sigma}_r$  where  $w = \sigma_1 \sigma_2 \dots \sigma_r$  with  $r = |w|, \sigma_j \in S$ ; note that, by a result of Tits,  $\dot{w}$  is well defined. Let  $N_0\mathbf{T}$  be the subgroup of  $N\mathbf{T}$  generated by  $\{\dot{\sigma}; \sigma \in S\}$ . This is a finite subgroup of  $N\mathbf{T}$  containing  $\dot{w}$  for any  $w \in W$ . Let  $\kappa_0 : N_0\mathbf{T} \rightarrow W$  be the restriction of  $\kappa : N\mathbf{T} \rightarrow W$ .

**1.3.** For  $n \in \mathbf{N}^*$  let  $\mathfrak{s}_n = \text{Hom}(\mathbf{T}_n, \bar{\mathbf{Q}}_l^*)$ ; we have  $\sharp(\mathfrak{s}_n) = n^\rho$ . For  $n, n'$  in  $\mathbf{N}^*$  such that  $n'/n \in \mathbf{Z}$ , the surjective homomorphism  $N_n^{n'} : \mathbf{T}_{n'} \rightarrow \mathbf{T}_n, t \mapsto t^{n'/n}$  induces an imbedding  $\mathfrak{s}_n \subset \mathfrak{s}_{n'}, \lambda \mapsto \lambda N_n^{n'}$ . Hence we can form the union  $\mathfrak{s}_\infty = \cup_{n \in \mathbf{N}^*} \mathfrak{s}_n$  (a countable abelian group). Then for any  $n \in \mathbf{N}^*$ ,  $\mathfrak{s}_n$  is a subgroup of  $\mathfrak{s}_\infty$ . Note also that  $\mathfrak{s}_\infty$  is the group of homomorphisms  $\mathbf{T}^\infty \rightarrow \bar{\mathbf{Q}}_l^*$  which factor through  $\mathbf{T}_n$  for some  $n \in \mathbf{N}^*$ . For any  $\lambda \in \mathfrak{s}_\infty$  there is a well defined local system  $L_\lambda$  on  $\mathbf{T}$  such

that for some/any  $n \in \mathbf{N}^*$  for which  $\lambda \in \mathfrak{s}_n$ ,  $L_\lambda$  is equivariant for the  $\mathbf{T}$ -action  $t_1 : t \mapsto t_1^n t$  on  $\mathbf{T}$  and the natural  $\mathbf{T}_n$  action the stalk of  $L_\lambda$  at 1 is through the character  $\lambda$ . For  $\lambda, \lambda' \in \mathfrak{s}_\infty$  we have canonically  $L_\lambda \otimes L_{\lambda'} = L_{\lambda\lambda'}$ ; for  $\lambda \in \mathfrak{s}_\infty$  we have canonically  $L_\lambda^* = L_{\lambda^{-1}}$ ; here  $()^*$  denotes the dual local system.

The  $W$ -action on  $\mathbf{T}$  restricts to a  $W$ -action on  $\mathbf{T}_n$  for any  $n \in \mathbf{N}^*$ . This induces a  $W$ -action on  $\mathbf{T}^\infty$ , a  $W$ -action on  $\mathfrak{s}_n$  for any  $n \in \mathbf{N}^*$ ; for  $\lambda \in \mathfrak{s}_n$ ,  $w \in W$  and  $t \in \mathbf{T}_n$  we have  $(w(\lambda))(t) = \lambda(w^{-1}(t))$ . There is a unique  $W$ -action of  $\mathfrak{s}_\infty$  which for any  $n \in \mathbf{N}^*$  restricts to the  $W$ -action on  $\mathfrak{s}_n$  just described. We set  $I = W \times \mathfrak{s}_\infty$ ; for  $w \in W, \lambda \in \mathfrak{s}_\infty$  we write  $w \cdot \lambda$  instead of  $(w, \lambda)$ .

**1.4.** If  $\alpha \in R$ , the coroot  $\check{\alpha} : \mathbf{k}^* \rightarrow \mathbf{T}$  restricts to a homomorphism  $\mathbf{k}_n^* \rightarrow \mathbf{T}_n$  for any  $n \in \mathbf{N}^*$  and by passage to projective limits, this induces a homomorphism  $\check{\alpha}^\infty : \mathbf{k}^\infty \rightarrow \mathbf{T}^\infty$  (notation of 0.2). Let  $\lambda \in \mathfrak{s}_\infty$ . We say that  $\alpha \in R_\lambda$  if the composition  $\mathbf{k}^\infty \xrightarrow{\check{\alpha}^\infty} \mathbf{T}^\infty \xrightarrow{\lambda} \bar{\mathbf{Q}}_l^*$  is identically 1 or equivalently if  $\check{\alpha}^* L_\lambda \cong \bar{\mathbf{Q}}_l$  as local systems on  $\mathbf{k}^*$ . Note that for  $w \in W$  we have  $w(R_\lambda) = R_{w(\lambda)}$ . Let  $R_\lambda^+ = R_\lambda \cap R^+$ ,  $R_\lambda^- = R_\lambda - R_\lambda^+$ . Let  $W_\lambda$  be the subgroup of  $W$  generated by  $\{\sigma_\alpha; \alpha \in R_\lambda\}$ . We have  $W_\lambda = W_{\lambda^{-1}}$ . Let  $W'_\lambda = \{w \in W; w(\lambda) = \lambda\}$ . We have  $W_\lambda \subset W'_\lambda$ . As in [L4, 5.3], there is a unique Coxeter group structure on  $W_\lambda$  with length function  $W_\lambda \rightarrow \mathbf{N}$ ,  $w \mapsto |w|_\lambda = \#\{\alpha \in R_\lambda^+; w(\alpha) \in R_\lambda^-\}$ ; note that, if  $w \in W_\lambda$  and  $w = \sigma_1 \sigma_2 \dots \sigma_r$  is any reduced expression of  $w$  in  $W$ , then

$$|w|_\lambda = \text{card}\{i \in [1, r]; \sigma_r \dots \sigma_{i+1} \sigma_i \sigma_{i+1} \dots \sigma_r \in W_\lambda\}.$$

**1.5.** For  $n \in \mathbf{N}^*$  we set  $I_n = \{w \cdot \lambda \in I; \lambda \in \mathfrak{s}_n\}$ . As in [L9, 31.2], let  $\mathbf{H}_n$  be the associative  $\mathcal{A}$ -algebra with generators  $T_w (w \in W)$ ,  $1_\lambda (\lambda \in \mathfrak{s}_n)$  and relations:

$$\begin{aligned} 1_\lambda 1_{\lambda'} &= \delta_{\lambda, \lambda'} 1_\lambda \text{ for } \lambda, \lambda' \in \mathfrak{s}_n; \\ T_w T_{w'} &= T_{ww'} \text{ if } w, w' \in W \text{ and } |ww'| = |w| + |w'|; \\ T_w 1_\lambda &= 1_{w(\lambda)} T_w \text{ for } w \in W, \lambda \in \mathfrak{s}_n; \\ T_\sigma^2 &= v^2 T_1 + (v^2 - 1) \sum_{\lambda \in \mathfrak{s}_n; \sigma \in W_\lambda} T_\sigma 1_\lambda \text{ for } \sigma \in S; \\ T_1 &= \sum_{\lambda \in \mathfrak{s}_n} 1_\lambda. \end{aligned}$$

The algebra  $\mathbf{H}_n$  is closely related to the algebra introduced by Yokonuma [Yo]. (It specializes to it under  $v = \sqrt{q}, n = q - 1$  where  $q$  is a power of a prime; this is shown in [L10, §35].) Note that  $T_1$  is the unit element of  $\mathbf{H}_n$ . In [L10, 31.2] it is shown that  $\{T_w 1_\lambda; w \cdot \lambda \in I_n\}$  is an  $\mathcal{A}$ -basis of  $\mathbf{H}_n$ . (In [L16, 1.7] we write  $\mathbf{H}$  instead of  $\mathbf{H}_n$ , but here we shall not do so.)

Now, for  $\sigma \in S$ ,  $T_\sigma$  is invertible in  $\mathbf{H}_n$ ; indeed, we have

$$T_\sigma^{-1} = v^{-2} T_\sigma + (1 - v^{-2}) \left( \sum_{\lambda \in \mathfrak{s}_n; \sigma \in W_\lambda} 1_\lambda \right).$$

It follows that  $T_w$  is invertible in  $\mathbf{H}_n$  for any  $w \in W$ . As shown in [L9, 31.3], there is a unique ring homomorphism  $\mathbf{H}_n \rightarrow \mathbf{H}_n$ ,  $h \mapsto \bar{h}$  such that  $\overline{T_w} = T_{w^{-1}}^{-1}$  for any  $w \in W$  and  $\overline{f 1_\lambda} = \bar{f} 1_\lambda$  for any  $f \in \mathcal{A}$ ,  $\lambda \in \mathfrak{s}_n$ . It is an involution called the *bar involution*.

If  $n, n' \in \mathbf{N}^*$  and  $n'/n \in \mathbf{Z}$ , then  $I_n \subset I_{n'}$  and the  $\mathcal{A}$ -linear map  $j_{n,n'} : \mathbf{H}_n \rightarrow \mathbf{H}_{n'}$  given by  $T_w 1_\lambda \mapsto T_w 1_\lambda$  for  $w \cdot \lambda \in I_n$  is an  $\mathcal{A}$ -algebra imbedding which does not necessarily preserve the unit element. Let  $\mathbf{H}$  be the union of all  $\mathbf{H}_n$  for various  $n \in \mathbf{N}^*$  according to the imbeddings  $j_{n,n'}$  above. Then  $\mathbf{H}$  is an  $\mathcal{A}$ -algebra without 1 in general; it has an  $\mathcal{A}$ -basis  $\{T_w 1_\lambda = 1_{w(\lambda)} T_w; w \cdot \lambda \in I\}$ . If  $n \in \mathbf{N}^*$ , then  $\mathbf{H}_n$  is the  $\mathcal{A}$ -submodule of  $\mathbf{H}$  with basis  $\{T_w 1_\lambda; w \cdot \lambda \in I_n\}$ ; it is an  $\mathcal{A}$ -subalgebra of  $\mathbf{H}$ . The algebra  $\mathbf{H}_n$  has been studied in [L10] and [L16, 1.7]. We shall often refer to *loc.cit.* for properties of  $\mathbf{H}$  which in *loc.cit.* are stated for  $\mathbf{H}_n$  with  $n$  fixed and which imply immediately the corresponding properties of  $\mathbf{H}$ .

We show that, if  $n, n' \in \mathbf{N}^*$  and  $n'/n \in \mathbf{Z}$ , then  $j_{n,n'} : \mathbf{H}_n \rightarrow \mathbf{H}_{n'}$  is compatible with the bar-involution on  $\mathbf{H}_n$  and  $\mathbf{H}_{n'}$ . It is enough to show that  $j_{n,n'}(\bar{\xi}) = \overline{j_{n,n'}(\xi)}$  for  $\xi = 1_\lambda, \lambda \in \mathfrak{s}_n$  or  $\xi = T_\sigma, \sigma \in S$ . The case where  $\xi = 1_\lambda, \lambda \in \mathfrak{s}_n$  is immediate. For  $\sigma \in S$  we have  $j_{n,n'}(T_\sigma) = T_\sigma \sum_{\lambda \in \mathfrak{s}_n} 1_\lambda$ , hence

$$\begin{aligned} j_{n,n'}(\overline{T_\sigma}) &= j_{n,n'}(v^{-2}T_\sigma + (1 - v^{-2})\left(\sum_{\lambda \in \mathfrak{s}_n; \sigma \in W_\lambda} 1_\lambda\right)) \\ &= v^{-2}T_\sigma \sum_{\lambda \in \mathfrak{s}_n} 1_\lambda + (1 - v^{-2})\left(\sum_{\lambda \in \mathfrak{s}_n; \sigma \in W_\lambda} 1_\lambda\right) = T_\sigma^{-1} \sum_{\lambda \in \mathfrak{s}_n} 1_\lambda = \overline{j_{n,n'}(T_\sigma)}, \end{aligned}$$

as desired. It follows that there is a unique ring homomorphism  $\mathbf{H} \rightarrow \mathbf{H}, h \mapsto \bar{h}$ , whose restriction to  $\mathbf{H}_n$  (for any  $n \in \mathbf{N}^*$ ) is the bar involution. This has square 1 and is again called the bar involution.

The  $\mathcal{A}$ -linear map  $\mathbf{H} \rightarrow \mathbf{H}, h \mapsto \tilde{h}$  given by  $T_w 1_\lambda \mapsto T_w 1_{\lambda^{-1}}$  for  $w \cdot \lambda \in I$  is an algebra involution. The  $\mathcal{A}$ -linear map  $\mathbf{H} \rightarrow \mathbf{H}, h \mapsto h^\flat$ , given by  $T_w 1_\lambda \mapsto 1_\lambda T_{w^{-1}}$  is an involutive algebra antiautomorphism. (See [L10, 32.19].)

**1.6.** As in [L10, 34.4], for any  $w \cdot \lambda \in I$  there is a unique element  $c_{w \cdot \lambda} \in \mathbf{H}$  such that

$$c_{w \cdot \lambda} = \sum_{y \in W} p_{y \cdot \lambda, w \cdot \lambda} v^{-|y|} T_y 1_\lambda$$

where  $p_{y \cdot \lambda, w \cdot \lambda} \in v^{-1}\mathbf{Z}[v^{-1}]$  if  $y \neq w$ ,  $p_{w \cdot \lambda, w \cdot \lambda} = 1$  and  $\overline{c_{w \cdot \lambda}} = c_{w \cdot \lambda}$ . For  $\lambda \in \mathfrak{s}_\infty$ ,  $y', w'$  in  $W_\lambda$  let  $P_{y', w'}^\lambda$  be the polynomial defined in [KL] in terms of the Coxeter group  $W_\lambda$ ; let

$$p_{y', w'}^\lambda = v^{-|w'|_\lambda + |y'|_\lambda} P_{y', w'}^\lambda(v^2) \in \mathbf{Z}[v^{-1}].$$

Let  $w \cdot \lambda \in I$ . From [L1, 1.9(i)] we see that  $wW_\lambda$  contains a unique element  $z$  such that  $|z|$  is minimum; we write  $z = \min(wW_\lambda)$ ; we have  $w = zw'$  with  $w' \in W_\lambda$ . We have

$$(a) \quad c_{w \cdot \lambda} = \sum_{y' \in W_\lambda} p_{y', w'}^\lambda v^{-|zy'|} T_{zy'} 1_\lambda.$$

See [L16, 1.8(a)]. From (a) we see that

$$p_{y \cdot \lambda, zw' \cdot \lambda} = p_{y', w'}^\lambda(v^2) \text{ if } y = zy', y' \in W_\lambda,$$

$$p_{y \cdot \lambda, zw' \cdot \lambda} = 0 \text{ if } y \notin zW_\lambda.$$

In particular we have  $p_{y \cdot \lambda, w \cdot \lambda} \in \mathbf{N}[v^{-1}]$ . From [L16, 1.8] for  $w \cdot \lambda \in I$  we have

$$\widetilde{c_{w \cdot \lambda}} = c_{w \cdot \lambda^{-1}}, c_{w \cdot \lambda}^b = c_{w^{-1} \cdot w(\lambda)}.$$

**1.7.** Now  $\mathbf{H}$  can be regarded as a two-sided ideal in an  $\mathcal{A}$ -algebra  $\mathbf{H}'$  with 1 as follows.

Let  $[\mathfrak{s}_\infty]$  be the set of formal  $\mathcal{A}$ -linear combinations  $\sum_{\lambda \in \mathfrak{s}_\infty} c_\lambda 1_\lambda$  with  $c_\lambda \in \mathcal{A}$ ; this is an  $\mathcal{A}$ -module in an obvious way. We regard  $[\mathfrak{s}_\infty]$  as a (commutative)  $\mathcal{A}$ -algebra with multiplication

$$\left( \sum_{\lambda \in \mathfrak{s}_\infty} c_\lambda 1_\lambda \right) \left( \sum_{\lambda \in \mathfrak{s}_\infty} c'_\lambda 1_\lambda \right) = \sum_{\lambda \in \mathfrak{s}_\infty} c_\lambda c'_\lambda 1_\lambda.$$

This algebra has a unit element  $1 = \sum_{\lambda \in \mathfrak{s}_\infty} 1_\lambda$ .

Let  $\mathbf{H}'$  be the  $\mathcal{A}$ -algebra with generators  $T_w (w \in W)$  and  $\phi \in [\mathfrak{s}_\infty]$  and relations:

$$T_w T_{w'} = T_{ww'} \text{ if } w, w' \in W \text{ and } |ww'| = |w| + |w'|;$$

$$T_\sigma^2 = v^2 T_1 + (v^2 - 1) T_\sigma \left( \sum_{\lambda \in \mathfrak{s}_\infty; \sigma \in W_\lambda} 1_\lambda \right) \text{ for } \sigma \in S;$$

$$T_w \phi = \phi' T_w \text{ for } \phi = \sum_{\lambda \in \mathfrak{s}_\infty} c_\lambda 1_\lambda, \phi' = \sum_{\lambda \in \mathfrak{s}_\infty} c_{w^{-1}(\lambda)} 1_\lambda \text{ in } [\mathfrak{s}_\infty], w \in W;$$

the map  $[\mathfrak{s}_\infty] \rightarrow \mathbf{H}'$ ,  $\xi \mapsto \xi$  respects the algebra structures.

It follows that  $\mathbf{H}'$  is a free left  $[\mathfrak{s}_\infty]$ -module with basis  $\{T_w; w \in W\}$  and a right free  $[\mathfrak{s}_\infty]$ -module with basis  $\{T_w; w \in W\}$ . Note that the algebra  $\mathbf{H}'$  has a unit element  $\sum_{\lambda \in \mathfrak{s}_\infty} 1_\lambda$ . Now  $\mathbf{H}$  can be identified with the two-sided ideal of  $\mathbf{H}'$  which as an  $\mathcal{A}$ -submodule is free with basis  $\{T_w 1_\lambda = 1_{w(\lambda)} T_w; w \cdot \lambda \in I\}$ .

**1.8.** Let  $W \setminus \mathfrak{s}_\infty$  be the set of  $W$ -orbits on  $\mathfrak{s}_\infty$ . For any  $\mathfrak{o} \in W \setminus \mathfrak{s}_\infty$  we set  $I_\mathfrak{o} = \{w \cdot \lambda \in I; \lambda \in \mathfrak{o}\}$ . This is a finite set. We have  $I = \sqcup_\mathfrak{o} I_\mathfrak{o}$ ,  $\mathbf{H} = \oplus_\mathfrak{o} \mathbf{H}_\mathfrak{o}$  where  $\mathbf{H}_\mathfrak{o}$  is the  $\mathcal{A}$ -submodule of  $\mathbf{H}$  spanned by  $\{T_w 1_\lambda = 1_{w(\lambda)} T_w; w \cdot \lambda \in I_\mathfrak{o}\}$  (thus,  $\mathbf{H}_\mathfrak{o}$  is a free  $\mathcal{A}$ -module of finite rank). If  $\mathfrak{o}, \mathfrak{o}'$  are distinct in  $W \setminus \mathfrak{s}_\infty$ , then clearly  $\mathbf{H}_\mathfrak{o} \mathbf{H}_{\mathfrak{o}'} = 0$ . Thus, each  $\mathbf{H}_\mathfrak{o}$  is a subalgebra of  $\mathbf{H}$ ; unlike  $\mathbf{H}$ , it has a unit element  $\sum_{\lambda \in \mathfrak{o}} 1_\lambda$ . It is stable under  $h \mapsto \bar{h}$  and under  $h \mapsto h^b$ . Moreover,  $h \mapsto \tilde{h}$  is an isomorphism of  $\mathbf{H}_\mathfrak{o}$  onto  $\mathbf{H}_{\mathfrak{o}^{-1}}$ . For any  $w \cdot \lambda \in I_\mathfrak{o}$  we have  $c_{w \cdot \lambda} \in \mathbf{H}_\mathfrak{o}$ ; moreover,  $\{c_{w \cdot \lambda}; w \cdot \lambda \in I_\mathfrak{o}\}$  is an  $\mathcal{A}$ -basis of  $\mathbf{H}_\mathfrak{o}$ .

**1.9.** For  $i, i'$  in  $I$  we write  $c_i c_{i'} = \sum_{j \in I} h_{i, i', j} c_j$  (product in  $\mathbf{H}$ ) where  $h_{i, i', j} \in \mathcal{A}$ . Let  $j \preceq_{\text{left}} i$  (resp.  $j \preceq i$ ) be the preorder on  $I$  generated by the relations  $h_{i', i, j} \neq 0$  for some  $i' \in I$ , resp. by the relations

$$h_{i, i', j} \neq 0 \text{ or } h_{i', i, j} \neq 0 \text{ for some } i' \in I.$$

We say that  $i \sim_{\text{left}} j$  (resp.  $i \sim j$ ) if  $i \preceq_{\text{left}} j$  and  $j \preceq_{\text{left}} i$  (resp.  $i \preceq j$  and  $j \preceq i$ ). This is an equivalence relation on  $I$ ; the equivalence classes are called left cells (resp. two-sided cells). Note that any two-sided cell is a union of left cells. Since for

$\mathfrak{o} \in W \setminus \mathfrak{s}_\infty$ ,  $\mathbf{H}_\mathfrak{o}$  is closed under left and right multiplication by elements in  $\mathbf{H}$ , we see that

$$h_{i,i',j} \neq 0, i \in I_\mathfrak{o} \text{ implies } i', j \in I_\mathfrak{o}; h_{i,i',j} \neq 0, i' \in I_\mathfrak{o} \text{ implies } i, j \in I_\mathfrak{o}.$$

It follows that  $j \preceq i, i \in I_\mathfrak{o}$  implies  $j \in I_\mathfrak{o}$ . In particular,  $j \sim i, i \in I_\mathfrak{o}$  implies  $j \in I_\mathfrak{o}$ . Thus any two-sided cell is contained in  $I_\mathfrak{o}$  for a unique  $\mathfrak{o}$ .

For  $i = w \cdot \lambda \in I$  we set

$$i^! = w^{-1} \cdot w(\lambda) \in I.$$

Note that  $i \mapsto i^!$  is an involution of  $I$  preserving  $I_\mathfrak{o}$  for any  $\mathfrak{o}$ .

If  $\mathbf{c}$  is a two-sided cell and  $i \in I$ , we write  $i \preceq \mathbf{c}$  (resp.  $\mathbf{c} \preceq i$ ) if  $i \preceq i'$  (resp.  $i' \preceq i$ ) for some  $i' \in \mathbf{c}$ ; we write  $i \prec \mathbf{c}$  (resp.  $\mathbf{c} \prec i$ ) if  $i \preceq \mathbf{c}$  (resp.  $\mathbf{c} \preceq i$ ) and  $i \notin \mathbf{c}$ . If  $\mathbf{c}, \mathbf{c}'$  are two-sided cells, we write  $\mathbf{c} \preceq \mathbf{c}'$  (resp.  $\mathbf{c} \prec \mathbf{c}'$ ) if  $i \preceq i'$  (resp.  $i \preceq i'$  and  $i \not\sim i'$ ) for some  $i \in \mathbf{c}, i' \in \mathbf{c}'$ .

Let  $j \in I$ . We can find an integer  $m \geq 0$  such that  $h_{i,i',j} \in v^{-m}\mathbf{Z}[v]$  for all  $i, i'$ ; let  $a(j)$  be the smallest such  $m$ . For  $i, i', j$  in  $I$  there is a well defined integer  $h_{i,i',j}^*$  such that

$$h_{i,i',j^!} = h_{i,i',j}^* v^{-a(j^!)} + \text{higher powers of } v.$$

Note that

$$h_{i,i',j}^* \neq 0, i \in I_\mathfrak{o} \text{ implies } i', j \in I_\mathfrak{o}; h_{i,i',j}^* \neq 0, i' \in I_\mathfrak{o} \text{ implies } i, j \in I_\mathfrak{o}.$$

Let  $\mathbf{D}$  be the set of all  $w \cdot \lambda \in I$  where  $w$  is a distinguished involution of the Coxeter group  $W_\lambda$ , see [L3]. We have  $\mathbf{D} = \sqcup_\mathfrak{o} (\mathbf{D} \cap \mathfrak{o})$ .

By [L16, 1.11], the following properties hold:

- Q1. If  $j \in \mathbf{D}$  and  $i, i' \in I$  satisfy  $h_{i,i',j}^* \neq 0$  then  $i' = i^*$ .
- Q2. If  $i \in I$ , there exists a unique  $j \in \mathbf{D}$  such that  $h_{i^!,i,j}^* \neq 0$ .
- Q3. If  $i' \preceq i$  then  $a(i') \geq a(i)$ . Hence if  $i' \sim i$  then  $a(i') = a(i)$ .
- Q4. If  $j \in \mathbf{D}$ ,  $i \in I$  and  $h_{i^!,i,j}^* \neq 0$  then  $h_{i^!,i,j}^* = 1$ .
- Q5. For any  $i, j, k$  in  $I$  we have  $h_{i,j,k}^* = h_{j,k,i}^*$ .
- Q6. Let  $i, j, k$  in  $I$  be such that  $h_{i,j,k}^* \neq 0$ . Then  $i \underset{\text{left}}{\sim} j^!, j \underset{\text{left}}{\sim} k^!, k \underset{\text{left}}{\sim} i^!$ .
- Q7. If  $i' \underset{\text{left}}{\preceq} i$  and  $a(i') = a(i)$  then  $i' \underset{\text{left}}{\sim} i$ .
- Q8. If  $i' \preceq i$  and  $a(i') = a(i)$  then  $i' \sim i$ .
- Q9. Any left cell  $\Gamma$  of  $I$  contains a unique element of  $j \in \mathbf{D}$ . We have  $h_{i^!,i,j}^* = 1$  for all  $i \in \Gamma$ .
- Q10. For any  $i \in I$  we have  $i \sim i^!$ .

Note that  $h_{i,j,k}^* \in \mathbf{N}$  for all  $i, j, k$  in  $I$ , see [L16, 1.11].

Let  $\mathbf{H}^\infty$  be the free abelian group with basis  $\{t_i; i \in I\}$ . We define a  $\mathbf{Z}$ -bilinear multiplication  $\mathfrak{A}^\infty \times \mathfrak{A}^\infty \rightarrow \mathfrak{A}^\infty$  by

$$t_i t_{i'} = \sum_{j \in I} h_{i,i',j^!}^* t_j.$$



For any  $\mathfrak{o} \in W \setminus \mathfrak{s}_\infty$  let  $\mathbf{H}_\mathfrak{o}^\infty$  be the free abelian subgroup of  $\mathbf{H}^\infty$  with basis  $\{t_i; i \in I_\mathfrak{o}\}$ . We have  $\mathbf{H}^\infty = \bigoplus_\mathfrak{o} \mathbf{H}_\mathfrak{o}^\infty$ ; moreover, if  $\mathfrak{o}, \mathfrak{o}'$  are distinct in  $W \setminus \mathfrak{s}_\infty$ , then  $\mathbf{H}_\mathfrak{o}^\infty \mathbf{H}_{\mathfrak{o}'}^\infty = 0$ . Thus each  $\mathbf{H}_\mathfrak{o}^\infty$  is a subalgebra of  $\mathbf{H}$ ; unlike  $\mathbf{H}^\infty$ ,  $\mathbf{H}_\mathfrak{o}^\infty$  has a unit element  $\sum_{i \in \mathbf{D} \cap \mathfrak{o}} t_i$ . The  $\mathbf{Z}$ -linear map  $\mathbf{H}^\infty \rightarrow \mathbf{H}^\infty$ ,  $h \mapsto h^b$  defined by  $t_i^b = t_{i^!}$  for all  $i \in I$  is a ring antiautomorphism preserving each  $\mathbf{H}_\mathfrak{o}^\infty$ . We define an  $\mathcal{A}$ -linear map  $\psi : \mathbf{H} \rightarrow \mathcal{A} \otimes \mathbf{H}^\infty$  by

$$\psi(c_i) = \sum_{i' \in I, j \in \mathbf{D}; i' \sim j} h_{i,j,i'} t_{i'} \text{ for all } i \in I.$$

(This last sum is finite. We have  $i \in I_\mathfrak{o}$  for some  $\mathfrak{o}$ . If  $h_{i,j,i'} \neq 0$  then we have  $i' \in \mathfrak{o}, j \in \mathfrak{o}$ . Thus  $i', j$  run through a finite set.) By [L16, 1.9, 1.11(vi)],  $\psi$  is a homomorphism of  $\mathcal{A}$ -algebras. For any  $\mathfrak{o}$ ,  $\psi$  restricts to a homomorphism of  $\mathcal{A}$ -algebras  $\psi_\mathfrak{o} : \mathbf{H}_\mathfrak{o} \rightarrow \mathcal{A} \otimes \mathbf{H}_\mathfrak{o}^\infty$  which takes 1 to 1.

We set  $\mathbf{H}^v = \mathbf{Q}(v) \otimes_\mathcal{A} \mathbf{H}$ ,  $\mathbf{J} = \mathbf{Q} \otimes \mathbf{H}^\infty$ ; for any  $\mathfrak{o}$  we set  $\mathbf{H}_\mathfrak{o}^v = \mathbf{Q}(v) \otimes_\mathcal{A} \mathbf{H}_\mathfrak{o}$ ,  $\mathbf{J}_\mathfrak{o} = \mathbf{Q} \otimes_\mathcal{A} \mathbf{H}_\mathfrak{o}^\infty$ . For any  $\mathfrak{o}$ ,  $\psi$  induces an algebra isomorphism  $\psi_\mathfrak{o}^v : \mathbf{H}_\mathfrak{o}^v \xrightarrow{\sim} \bar{\mathbf{Q}}_l(v) \otimes \mathbf{J}_\mathfrak{o}$ ; hence  $\psi$  induces an algebra isomorphism  $\psi^v : \mathbf{H}^v \xrightarrow{\sim} \bar{\mathbf{Q}}_l(v) \otimes \mathbf{J}$ .

We define a group homomorphism  $\mathbf{t} : \mathbf{H}^\infty \rightarrow \mathbf{Z}$  by  $\mathbf{t}(t_i) = 1$  if  $i \in \mathbf{D}$ ,  $\mathbf{t}(t_i) = 0$  if  $i \in I - \mathbf{D}$ . As in [L16, 1.9(a)], the following can be deduced from Q1, Q2, Q4.

(a) For  $i, j \in I$  we have  $\mathbf{t}(t_i t_j) = 1$  if  $j = i^!$  and  $\mathbf{t}(t_i t_j) = 0$  if  $j \neq i^!$ .

**1.10.** For  $n \in \mathbf{N}^*$  we set  $\mathbf{H}_n^1 = \bar{\mathbf{Q}}_l \otimes_\mathcal{A} \mathbf{H}_n$ ; this is a  $\bar{\mathbf{Q}}_l$ -algebra with 1. It is the algebra with generators  $T_w (w \in W)$ ,  $1_\lambda (\lambda \in \mathfrak{s}_n)$  and relations:

$$\begin{aligned} 1_\lambda 1_{\lambda'} &= \delta_{\lambda, \lambda'} 1_\lambda \text{ for } \lambda, \lambda' \in \mathfrak{s}_n; \\ T_w T_{w'} &= T_{ww'} \text{ for } w, w' \in W; \\ T_w 1_\lambda &= 1_{w(\lambda)} T_w \text{ for } w \in W, \lambda \in \mathfrak{s}_n; \\ T_1 &= \sum_{\lambda \in \mathfrak{s}_n} 1_\lambda. \end{aligned}$$

It has a basis  $\{T_w 1_\lambda; w \cdot \lambda \in I_n\}$ . Let  $\mathbf{H}^1 = \bar{\mathbf{Q}}_l \otimes_\mathcal{A} \mathbf{H}$ . This is a  $\bar{\mathbf{Q}}_l$ -algebra without 1 in general. As a vector space it has basis  $\{T_w 1_\lambda, w \cdot \lambda \in I\}$ . It contains naturally  $\mathbf{H}_n^1$  as a subalgebra for any  $n \in \mathbf{N}^*$ . For any  $\mathfrak{o} \in W \setminus \mathfrak{s}_\infty$  we set  $\mathbf{H}_\mathfrak{o}^1 = \bar{\mathbf{Q}}_l \otimes_\mathcal{A} \mathbf{H}_\mathfrak{o}$ ; this is a  $\bar{\mathbf{Q}}_l$ -algebra with 1. It has a basis  $\{T_w 1_\lambda; w \cdot \lambda \in I_\mathfrak{o}\}$ . We have  $\mathbf{H}^1 = \bigoplus_\mathfrak{o} \mathbf{H}_\mathfrak{o}^1$ . Now  $\psi$  in 1.9 induces an algebra isomorphism  $\psi^1 : \mathbf{H}^1 \xrightarrow{\sim} \mathbf{J}$ ; for any  $\mathfrak{o}$ ,  $\psi_\mathfrak{o}$  in 1.9 induces an algebra isomorphism  $\psi_\mathfrak{o}^1 : \mathbf{H}_\mathfrak{o}^1 \xrightarrow{\sim} \mathbf{J}_\mathfrak{o}$  taking 1 to 1.

**1.11.** Let  $n \in \mathbf{N}^*$ . Consider the group algebra  $\bar{\mathbf{Q}}_l[W\mathbf{T}_n]$  where  $W\mathbf{T}_n$  is the semidirect product of  $W$  and  $\mathbf{T}_n$  with  $\mathbf{T}_n$  normal and  $W$  acting on  $\mathbf{T}_n$  by  $w : t \mapsto w(t)$ . Now  $wt \mapsto \sum_{\lambda \in \mathfrak{s}_n} \lambda(t) T_w 1_\lambda$  defines a  $\bar{\mathbf{Q}}_l$ -linear isomorphism  $u_n : \bar{\mathbf{Q}}_l[W\mathbf{T}_n] \xrightarrow{\sim} \mathbf{H}_n^1$  which is in fact an algebra isomorphism taking 1 to 1.

Now let  $n, n' \in \mathbf{N}^*$  be such that  $n'/n \in \mathbf{Z}$ . We define a  $\bar{\mathbf{Q}}_l$ -linear imbedding  $h_{n,n'} : \bar{\mathbf{Q}}_l[W\mathbf{T}_n] \rightarrow \bar{\mathbf{Q}}_l[W\mathbf{T}_{n'}]$  by

$$h_{n,n'}(wt) = (n/n')^\rho \sum_{t' \in \mathbf{T}_{n'}; t'^{n'/n} = t} wt'.$$

We show that  $h_{n,n'}$  is compatible with multiplication, that is, for  $w, w'$  in  $W$  and  $t, t'$  in  $\mathbf{T}_n$  we have

$$\begin{aligned} & ((n/n')^\rho \sum_{\tilde{t} \in \mathbf{T}_{n'}; \tilde{t}^{n'/n} = t} w\tilde{t}) ((n/n')^\rho \sum_{\tilde{t}' \in \mathbf{T}_{n'}; \tilde{t}'^{n'/n} = t'} w'\tilde{t}') \\ &= (n/n')^\rho \sum_{\tilde{t}'' \in \mathbf{T}_{n'}; \tilde{t}''^{n'/n} = w'^{-1}(t)t'} ww'\tilde{t}'', \end{aligned}$$

or equivalently

$$((n/n')^\rho \sum_{\tilde{t}, \tilde{t}' \in \mathbf{T}_{n'}; \tilde{t}^{n'/n} = t, \tilde{t}'^{n'/n} = t'} w'^{-1}(\tilde{t})\tilde{t}') \sum_{\tilde{t}'' \in \mathbf{T}_{n'}; \tilde{t}''^{n'/n} = w'^{-1}(t)t'} \tilde{t}'',$$

which is easily verified.

Let  $j_{n,n'}^1 : \mathbf{H}_n^1 \xrightarrow{\sim} \mathbf{H}_{n'}^1$  be the specialization of  $j_{n,n'}$  (see 1.5) at  $v = 1$ . We have  $u_{n'} h_{n,n'} = j_{n,n'} u_n$ ; equivalently for  $w \in W, t \in \mathbf{T}_n$ , we have

$$(n/n')^\rho \sum_{t' \in \mathbf{T}_{n'}; t'^{n'/n} = t} \sum_{\lambda \in \mathfrak{s}_{n'}} \lambda(t') T_w 1_\lambda = \sum_{\lambda \in \mathfrak{s}_n} \lambda(t) T_w 1_\lambda.$$

(It is enough to show that for any  $\lambda \in \mathfrak{s}_{n'}$ ,

$$(n/n')^\rho \sum_{t' \in \mathbf{T}_{n'}; t'^{n'/n} = t} \lambda(t') = \lambda(t).$$

is equal to  $\lambda(t)$  if  $\lambda \in \mathfrak{s}_n$  and to 0 if  $\lambda \notin \mathfrak{s}_n$ . This is immediate: we use that the kernel of the surjective homomorphism  $\mathbf{T}_{n'} \rightarrow \mathbf{T}_n, t' \mapsto t'^{n'/n}$  has exactly  $(n'/n)^\rho$  elements.)

We can form the union  $\cup_{n \in \mathbf{N}^*} \bar{\mathbf{Q}}_l[W\mathbf{T}_n]$  over all imbeddings  $h_{n,n'}$  as above. This union has an algebra structure whose restriction to  $\bar{\mathbf{Q}}_l[W\mathbf{T}_n]$  (for any  $n \in \mathbf{N}^*$ ) is the algebra structure of  $\bar{\mathbf{Q}}_l[W\mathbf{T}_n]$ . Moreover, there is a unique isomorphism of algebras  $\cup_{n \in \mathbf{N}^*} \bar{\mathbf{Q}}_l[W\mathbf{T}_n] \xrightarrow{\sim} \mathbf{H}^1$  whose restriction to  $\bar{\mathbf{Q}}_l[W\mathbf{T}_n]$  (for any  $n \in \mathbf{N}^*$ ) is  $u_n : \bar{\mathbf{Q}}_l[W\mathbf{T}_n] \xrightarrow{\sim} \mathbf{H}_n^1$ .

**1.12.** For  $\mathfrak{o} \in W \setminus \mathfrak{s}_\infty$ ,  $\mathbf{H}_\mathfrak{o}^1$  is a semisimple  $\bar{\mathbf{Q}}_l$ -algebra. Let  $\text{Irr}(H_\mathfrak{o}^1)$  be a set of representatives for the isomorphism classes of simple  $\mathbf{H}_\mathfrak{o}^1$ -modules.

**1.13.** We have  $\mathbf{H}^\infty = \oplus_{\mathbf{c}} \mathbf{H}_\mathbf{c}^\infty$ ,  $\mathbf{J} = \oplus_{\mathbf{c}} \mathbf{J}_\mathbf{c}$ , where  $\mathbf{c}$  runs over the two-sided cells in  $I$ ,  $\mathbf{H}_\mathbf{c}^\infty$  is the  $\mathcal{A}$ -submodule of  $\mathbf{H}^\infty$  with basis  $\{t_i; i \in \mathbf{c}\}$  and  $\mathbf{J}_\mathbf{c}$  is the  $\bar{\mathbf{Q}}_l$ -subspace of  $\mathbf{J}$  with basis  $\{t_i; i \in \mathbf{c}\}$ . Each  $\mathbf{H}_\mathbf{c}^\infty$  is an  $\mathcal{A}$ -subalgebra of  $\mathbf{H}^\infty$  with unit  $\sum_{i \in \mathbf{D}_\mathbf{c}} t_i$  where  $\mathbf{D}_\mathbf{c} = \mathbf{D} \cap \mathbf{c}$ . Each  $\mathbf{J}_\mathbf{c}$  is a  $\bar{\mathbf{Q}}_l$ -subalgebra of  $\mathbf{J}$  with the same unit as  $\mathbf{H}_\mathbf{c}^\infty$ . Moreover if  $\mathbf{c}, \mathbf{c}'$  are distinct two-sided cells in  $I$  we have  $\mathbf{J}_\mathbf{c} \mathbf{J}_{\mathbf{c}'} = 0$ . Recall from 1.9 that any two-sided cell in  $I$  is contained in  $I_\mathfrak{o}$  for a unique  $\mathfrak{o} \in W \setminus \mathfrak{s}_\infty$ . It follows that for any  $\mathfrak{o} \in W \setminus \mathfrak{s}_\infty$  we have  $\mathbf{J}_\mathfrak{o} = \oplus_{\mathbf{c} \subset I_\mathfrak{o}} \mathbf{J}_\mathbf{c}$ . Hence, if  $E \in \text{Irr}(H_\mathfrak{o}^1)$  then there is a unique two-sided cell  $\mathbf{c}_E$  such that  $\mathbf{J}_\mathbf{c}$  acts as zero on  $E^\infty$  for any  $\mathbf{c} \subset I_\mathfrak{o}$  with  $\mathbf{c} \neq \mathbf{c}_E$ . Thus  $E^\infty$  can be viewed as a simple  $\mathbf{J}_{\mathbf{c}_E}$ -module. We define  $a_E \in \mathbf{N}$  to be the constant value of the restriction of  $a : I \rightarrow \mathbf{N}$  to  $\mathbf{c}_E$ .

**1.14.** If  $\mathbf{c}$  is a two-sided cell of  $I$  then its image  $\tilde{\mathbf{c}}$  under  $I \rightarrow I, w \cdot \lambda \mapsto w \cdot \lambda^{-1}$  is a two-sided cell of  $I$ . (See [L16, 1.14].) As noted in 1.9, we have  $\mathbf{c} \subset I_{\mathbf{o}}$  for a unique  $\mathbf{o}$ ; from the definitions we have  $\tilde{\mathbf{c}} \subset I_{\mathbf{o}^{-1}}$ . Moreover, the value of the  $a$ -function on  $\tilde{\mathbf{c}}$  is equal to the value of the  $a$ -function on  $\mathbf{c}$ . From Q3, Q10 in 1.9, we see that  $a(i^!) = a(i)$  for  $i \in I$ .

**1.15.** For  $i, i'$  in  $I$  we show:

- (a) If  $i \underset{\text{left}}{\sim} i'$ , then for some  $u \in I$ ,  $t_{i'}$  appears with  $\neq 0$  coefficient in  $t_u t_i$ .
- (b) If  $i^! \underset{\text{left}}{\sim} i'^!$ , then for some  $u \in I$ ,  $t_{i'}$  appears with  $\neq 0$  coefficient in  $t_i t_u$ .
- (c) If  $i \underset{\text{left}}{\sim} i'$ , then for some  $u, u'$  in  $I$ ,  $t_{i'}$  appears with nonzero coefficient in  $t_u t_i t_{u'}$ .
- (d) If  $i \sim i'$ , then  $t_i t_j t_{i'} \neq 0$  for some  $j \in I$ .

The proof is along the lines of that of [L8, 18.4]. Let  $J^+ = \sum_{k \in I} \mathbf{N} t_k$ . We will use repeatedly that  $J^+ J^+ \subset J^+$ .

Let  $i, i'$  be as in (a). Let  $d, d' \in \mathbf{D}$  be such that  $h_{i^!, i, d}^* \neq 0$  and  $h_{i'^!, i', d'}^* \neq 0$ . Then  $i \underset{\text{left}}{\sim} d, i' \underset{\text{left}}{\sim} d'$ . Hence  $d \underset{\text{left}}{\sim} d'$ . By Q9 in 1.9 we have  $d = d'$  and  $h_{i^!, i, d}^* = 1, h_{i'^!, i', d}^* = 1$ . Hence  $t_{i'} t_i = t_d + J^+, t_{i'} t_{i'} = t_d + J^+, t_d t_d = t_d$ ; it follows that  $t_{i'} t_i t_{i'} \in t_d t_d + J^+ = t_d + J^+$ . In particular,  $t_i t_{i'} \neq 0$ . Thus,  $h_{i^!, i', u}^* \neq 0$  for some  $u \in I$ . Using Q5 in 1.9 we deduce that  $h_{u, i, i'}^* \neq 0$  hence  $t_{i'}$  appears with  $\neq 0$  coefficient in  $t_u t_i$ . This proves (a). Now (b) follows from (a) using the antiautomorphism of  $\mathbf{H}^\infty$  such that  $t_u \mapsto t_{u'}$  for all  $u \in I$ .

Let  $i_1, i_2, i_3$  in  $I$  be such that  $i_1 \sim i_2 \sim i_3$ . If the conclusion of (c) holds for  $(i, i') = (i_1, i_2)$  and for  $(i, i') = (i_2, i_3)$  then clearly it holds for  $(i, i') = (i_1, i_3)$ . Applying this repeatedly, we see that it is enough to prove (c) in the case where  $i, i'$  satisfy either  $i \underset{\text{left}}{\sim} i'$  or  $i^! \underset{\text{left}}{\sim} i'^!$ . In these cases the desired result follows from (a), (b).

Let  $i, i'$  be as in (d). Then  $i \sim i'$ . By (c), we have  $t_u t_i t_u \in a t_{i'} + J^+$  for some  $u, u' \in I$  and some  $a \in \mathbf{Z}_{>0}$ . Hence  $t_u t_i t_u t_{i'} \in a t_{i'} t_{i'} + J^+$ . Since  $t_{i'} t_{i'}$  has some coefficient 1 and the other coefficients are  $\geq 0$ , it follows that  $t_u t_i t_u t_{i'} \neq 0$ . Thus,  $t_i t_u t_{i'} \neq 0$ . This proves (d).

## 2. THE GROUP $\tilde{G}$

**2.1.** In this paper (except in 2.2) we fix a group  $\tilde{G}$  containing  $G$  as a subgroup, such that  $\tilde{G}/G$  is cyclic of order  $\mathbf{m} \leq \infty$  with a fixed generator. For  $s \in \mathbf{Z}$  let  $\tilde{G}_s$  be the inverse image of the  $s$ -th power of this generator under the obvious map  $\tilde{G} \rightarrow \tilde{G}/G$ . For  $\gamma \in \tilde{G}$ , the map  $G \rightarrow G, g \mapsto \gamma g \gamma^{-1}$  is denoted by  $\text{Ad}(\gamma)$ .

We shall always assume that we are in one of the two cases below (later referred to as case A and case B).

(A) We have  $\mathbf{m} = \infty$  and one of the following two equivalent conditions are satisfied ( $q$  denotes a fixed power of  $p$ ):

(i) for some  $\gamma \in \tilde{G}_1$ ,  $\text{Ad}(\gamma) : G \rightarrow G$  is the Frobenius map for an  $F_q$ -rational structure on  $G$ ;

(ii) for any  $s > 0$  and any  $\gamma \in \tilde{G}_s$ ,  $\text{Ad}(\gamma) : G \rightarrow G$  is the Frobenius map for an  $F_{q^s}$ -rational structure on  $G$ .

(B)  $\mathbf{m} < \infty$  and  $\tilde{G}$  is an algebraic group with identity component  $G$ .

We show the equivalence of (i),(ii) in case A. Clearly, if (ii) holds then (i) holds. Conversely, assume that (i) holds for  $\gamma \in \tilde{G}_1$ . If  $\gamma' \in \tilde{G}_s$  with  $s > 0$ , then we have  $\gamma' = g_1 \gamma^s$  where  $g_1 \in G$ . By Lang's theorem applied to  $\text{Ad}(\gamma^s) : G \rightarrow G$ , which is the Frobenius map for an  $F_{q^s}$ -rational structure on  $G$ , we have  $g_1 = g_2^{-1} \text{Ad}(\gamma^s)(g_2)$  for some  $g_2 \in G$  hence  $\gamma' = g_2^{-1} \text{Ad}(\gamma^s)(g_2) \gamma^s = g_2^{-1} \gamma^s g_2$  and  $\text{Ad}(\gamma') = \text{Ad}(g_2)^{-1} \text{Ad}(\gamma^s) \text{Ad}(g_2)$ . Since  $\text{Ad}(g_2) : G \rightarrow G$  is an isomorphism of algebraic varieties, it follows that  $\text{Ad}(\gamma') : G \rightarrow G$  is the Frobenius map for an  $F_{q^s}$ -rational structure on  $G$ . Thus (ii) holds.

Let  $s \in \mathbf{Z}$ . In case B,  $\tilde{G}_s$  is naturally an algebraic variety. In case A, we view  $\tilde{G}_s$  as an algebraic variety using the bijection  $g \mapsto g\gamma$  where  $\gamma$  is fixed in  $\tilde{G}_s$ ; this algebraic structure on  $\tilde{G}_s$  is independent of the choice of  $\gamma$ . For  $s = 0$  this gives the usual structure of algebraic variety of  $G$ . For  $s \in \mathbf{Z}, s' \in \mathbf{Z}$ , the multiplication  $\tilde{G}_s \times \tilde{G}_{s'} \rightarrow \tilde{G}_{s+s'}$  is obviously a morphism of algebraic varieties in case B, but is only a quasi-morphism in the sense of [L15, 0.3] in case A. Similarly, for  $s \in \mathbf{Z}$ ,  $\tilde{G}_s \rightarrow \tilde{G}_{-s}, \gamma \mapsto \gamma^{-1}$  is a morphism of algebraic varieties in case B, but is only a quasi-morphism in case A.

Note that in case A with  $s \neq 0$ , the conjugation action of  $G$  on  $\tilde{G}_s$  is transitive. (If  $s > 0$ , this follows from as above using Lang's theorem, while if  $s < 0$  this follows using the bijection  $\tilde{G}_s \rightarrow \tilde{G}_{-s}, \gamma \mapsto \gamma^{-1}$ , which commutes with the  $G$ -actions.) Moreover in this case for any  $\gamma \in \tilde{G}_s$ , the stabilizer of  $\gamma$  for this  $G$ -action is finite. (This stabilizer is the fixed point set of  $\text{Ad}(\gamma) : G \rightarrow G$  which is a Frobenius map relative to an  $F_{q^s}$ -structure if  $s > 0$  or the inverse of a Frobenius map if  $s < 0$ .)

We show:

(a) If  $\gamma \in \tilde{G}_s$  and  $B \in \mathcal{B}$  then  $\text{Ad}(\gamma)(B) \in \mathcal{B}$ ,  $\text{Ad}(\gamma)(U_B) = U_{\text{Ad}(\gamma)B}$  and  $\text{Ad}(\gamma) : \mathcal{B} \rightarrow \mathcal{B}$  is a bijection.

In case A with  $s = 0$  and in case B, (a) is obvious. In case A with  $s > 0$ , (a) follows from (ii); in case A with  $s < 0$ , (a) follows from (ii) applied to  $\gamma^{-1}$ .

## 2.2. Here are some examples in case A.

(i) Let  $F : G \rightarrow G$  be the Frobenius map for an  $F_q$ -rational structure on  $G$ . Let  $\tilde{G} = G \times \mathbf{Z}$  regarded as a group with multiplication  $(g, s)(g', s') = (gF^s(g'), s + s')$ . Define a homomorphism  $\tilde{G} \rightarrow \mathbf{Z}$  by  $(g, s) \mapsto s$ . Its kernel  $\{(g, s) \in \tilde{G}; s = 0\}$  can be identified with  $G$ . Note that  $\tilde{G}$  and  $\tilde{G} \rightarrow \mathbf{Z}$  are as in case A; we have  $(1, 1) \in \tilde{G}_1$  and  $\text{Ad}(1, 1) : G \rightarrow G$  is just  $F : G \rightarrow G$ . Moreover, any  $\tilde{G}$  and  $\tilde{G} \rightarrow \mathbf{Z}$  as in case A is obtained by the procedure above.

(ii) In the case where  $G$  is adjoint we define  $\tilde{G}_s$  for  $s \in \mathbf{Z}_{<0}$  to be the set of Frobenius maps  $G \rightarrow G$  with respect to various split  $F_{q^s}$ -rational structures on  $G$ ; we define  $\tilde{G}_s$  for  $s \in \mathbf{Z}_{<0}$  to be the set of maps  $G \rightarrow G$  whose inverse is in

$\tilde{G}_{-s}$  and we set  $\tilde{G}_0 = G$ . Then  $\tilde{G} = \sqcup_{s \in \mathbf{Z}} \tilde{G}_s$  is as in case A. (This case has been considered in [L15].)

(iii) Let  $V$  be a finite dimensional  $\mathbf{k}$ -vector space. For any  $s \in \mathbf{Z}$  let  $\widetilde{GL(V)}_s$  be the set of all group isomorphisms  $T : V \rightarrow V$  such that  $T(zx) = z^{q^s} T(x)$  for all  $z \in \mathbf{k}, x \in V$ ; in particular we have  $\widetilde{GL(V)}_0 = GL(V)$ . Then  $\widetilde{GL(V)} := \sqcup_{s \in \mathbf{Z}} \widetilde{GL(V)}_s$  is a group under composition of maps; it is of the form  $\tilde{G}$  (as in case A) where  $G = GL(V)$ .

(iv) Let  $V$  be a finite dimensional  $\mathbf{k}$ -vector space with a nondegenerate symplectic form  $(, ) : V \times V \rightarrow \mathbf{k}$ . For any  $s \in \mathbf{Z}$  let  $\widetilde{Sp(V)}_s$  be the set of all  $T \in \widetilde{GL(V)}_s$  such that  $(T(x), T(x')) = (x, x')^{q^s}$  for all  $x, x'$  in  $V$ ; in particular we have  $\widetilde{Sp(V)}_0 = Sp(V)$ . Then  $\widetilde{Sp(V)} := \sqcup_{s \in \mathbf{Z}} \widetilde{Sp(V)}_s$  is a group under composition of maps; it is of the form  $\tilde{G}$  (as in case A) where  $G = Sp(V)$ .

**2.3.** *In the rest of this paper we fix  $\tau \in \tilde{G}_1$  such that  $\tau \mathbf{B} \tau^{-1} = \mathbf{B}, \tau \mathbf{T} \tau^{-1} = \mathbf{T}$ . and such that for any  $\sigma \in S$ ,  $\text{Ad}(\tau)$  carries  $\xi_\sigma \in \mathbf{U}_\sigma - \{1\}$  to  $\xi_{\sigma'} \in \mathbf{U}_{\sigma'} - \{1\}$  for some  $\sigma' \in S$ .*

Note that such  $\tau$  exists.

We define a group homomorphism  $\mathbf{e} : \tilde{G} \rightarrow \tilde{G}$  by  $\mathbf{e}(\gamma) = \tau \gamma \tau^{-1}$ . We have  $\mathbf{e}(\tilde{G}_s) = \tilde{G}_s$  for all  $s \in \mathbf{Z}$ ,  $\mathbf{e}(\mathbf{T}) = \mathbf{T}$ ,  $\mathbf{e}(\mathbf{B}) = \mathbf{B}$  (hence  $\mathbf{e}(\mathbf{U}) = \mathbf{U}$ ),  $\mathbf{e}(N\mathbf{T}) = N\mathbf{T}$ ; thus  $\mathbf{e}$  induces an automorphism of  $W$  denoted again by  $\mathbf{e}$  which preserves the Coxeter group structure. If  $B \in \mathcal{B}$  then  $\mathbf{e}(B) \in \mathcal{B}$  and  $B \mapsto \mathbf{e}(B)$ ,  $\mathcal{B} \rightarrow \mathcal{B}$  is an automorphism in case B and is the Frobenius map for an  $\mathbf{F}_q$ -rational structure on  $\mathcal{B}$  in case A. We define  $\mathbf{e} : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B}$  by  $\mathbf{e}(B, B') = (\mathbf{e}(B), \mathbf{e}(B'))$ . For  $w \in W$  we have  $\mathbf{e}(G_w) = G_{\mathbf{e}(w)}$  and  $\mathbf{e}(\mathcal{O}_w) = \mathcal{O}_{\mathbf{e}(w)}$ .

The set  $\{\dot{\sigma}; \sigma \in S\}$  of  $N\mathbf{T}$  is stable under  $\mathbf{e} : N\mathbf{T} \rightarrow N\mathbf{T}$ . For  $w \in W$  we have  $(\mathbf{e}(w))^\cdot = \mathbf{e}(w^\cdot)$ . Hence  $N_0\mathbf{T}$  is stable under  $\mathbf{e} : N\mathbf{T} \rightarrow N\mathbf{T}$ .

Now for  $n \in \mathbf{N}^*$ ,  $\mathbf{e} : \mathbf{T} \rightarrow \mathbf{T}$  restricts to an isomorphism  $\mathbf{e} : \mathbf{T}_n \rightarrow \mathbf{T}_n$  and this induces an isomorphism  $\mathbf{e} : \mathfrak{s}_n \rightarrow \mathfrak{s}_n$  by  $\lambda \mapsto \mathbf{e}(\lambda)$  where  $(\mathbf{e}(\lambda))(t) = \lambda(\mathbf{e}^{-1}(t))$  for  $t \in \mathbf{T}_n$ . Let  $\mathbf{e} : \mathfrak{s}_\infty \rightarrow \mathfrak{s}_\infty$  be the isomorphism whose restriction to  $\mathfrak{s}_n$  is  $\mathbf{e} : \mathfrak{s}_n \rightarrow \mathfrak{s}_n$  as above for any  $n \in \mathbf{N}^*$ .

We shall fix a Frobenius map  $\Psi : G \rightarrow G$  relative to some sufficiently large finite subfield  $F_{q'}$  of  $\mathbf{k}$  such that  $\mathbf{B}, \mathbf{T}$  are  $\Psi$ -stable,  $\Psi$  acts on  $t$  by  $t \mapsto t^{q'}$  (hence it acts as the identity on  $W$ ) and such that  $\Psi \mathbf{e} = \mathbf{e} \Psi : G \rightarrow G$  and  $\Psi(\omega) = \omega$  for any  $\omega \in N_0\mathbf{T}$ ; in case B we also require that  $\Psi(\tau^{\mathbf{m}}) = \tau^{\mathbf{m}}$ .

For any  $s \in \mathbf{Z}$  we define an  $F_{q'}$ -rational structure on  $\tilde{G}_s$  with Frobenius map  $\Psi : \tilde{G}_s \rightarrow \tilde{G}_s$  by the requirement that  $\Psi(g\tau^s) = \Psi(g)\tau^s$  for any  $g \in G$ ; in case B, this rational structure depends only on  $\tilde{G}_s$  not on  $s$ .

Now for any  $n \in \mathbf{N}^*$  we have  $\Psi(\mathbf{T}_n) = \mathbf{T}_n$ ; hence we can define  $\Psi : \mathfrak{s}_n \xrightarrow{\sim} \mathfrak{s}_n$  by  $(\Psi\lambda)(t) = \lambda(\Psi^{-1}(t))$  for  $t \in \mathbf{T}_n$ ,  $\lambda \in \mathfrak{s}_n$ . There is a unique bijection  $\Psi : \mathfrak{s}_\infty \rightarrow \mathfrak{s}_\infty$  whose restriction to  $\mathfrak{s}_n$  is as above for any  $n \in \mathbf{N}^*$ . Now  $\Psi$  induces  $F_{q'}$ -rational

structures on various varieties that will appear in the sequel. When we consider  $\mathcal{D}_m()$  or  $\mathcal{M}_m()$  for such varieties, we will refer to these specific  $F_{q'}$ -structures.

**2.4.** We define a bijection  $\mathbf{e} : I \rightarrow I$  by  $\mathbf{e}(w \cdot \lambda) = \mathbf{e}(w) \cdot \mathbf{e}(\lambda)$ . The  $\mathcal{A}$ -linear map  $\mathbf{e} : \mathbf{H} \rightarrow \mathbf{H}$  defined by  $\mathbf{e}(T_w 1_\lambda) = T_{\mathbf{e}(w)} 1_{\mathbf{e}(\lambda)}$  for  $w \cdot \lambda \in I$  is an algebra isomorphism commuting with  $\bar{\cdot} : \mathbf{H} \rightarrow \mathbf{H}$ . It follows that  $\mathbf{e}(c_i) = c_{\mathbf{e}(i)}$  for all  $i \in I$  and that  $\mathbf{e} : I \rightarrow I$  maps any left (resp. two-sided) cell of  $I$  onto a left (resp. two-sided) cell of  $I$ . It also maps any  $W$ -orbit in  $\mathfrak{s}_\infty$  onto a  $W$ -orbit in  $\mathfrak{s}_\infty$ .

Let  $\mathfrak{o} \in \mathfrak{s}_\infty$  and  $s \in \mathbf{Z}$  be such that  $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$ . The  $\mathcal{A}$ -linear map  $\mathbf{e}^s : \mathbf{H} \rightarrow \mathbf{H}$  restricts to an  $\mathcal{A}$ -algebra isomorphism  $\mathbf{e}^s : \mathbf{H}_\mathfrak{o} \rightarrow \mathbf{H}_\mathfrak{o}$ ; this gives rise by extension of scalars to a  $\bar{\mathbf{Q}}_l$ -algebra isomorphism  $\mathbf{e}^s : \mathbf{H}_\mathfrak{o}^1 \rightarrow \mathbf{H}_\mathfrak{o}^1$  and to a  $\bar{\mathbf{Q}}_l(v)$ -algebra isomorphism  $\mathbf{e} : \mathbf{H}_\mathfrak{o}^v \rightarrow \mathbf{H}_\mathfrak{o}^v$ ; moreover the  $\bar{\mathbf{Q}}_l$ -linear map  $\mathbf{e}^s : \mathbf{J}_\mathfrak{o} \rightarrow \mathbf{J}_\mathfrak{o}$  given by  $t_i \mapsto t_{\mathbf{e}^s(i)}$  for  $i \in I_\mathfrak{o}$  is an algebra isomorphism and  $\psi_\mathfrak{o}^v : \mathbf{H}_\mathfrak{o}^v \xrightarrow{\sim} \bar{\mathbf{Q}}_l(v) \otimes \mathbf{J}_\mathfrak{o}$ ,  $\psi_\mathfrak{o}^1 : \mathbf{H}_\mathfrak{o}^1 \xrightarrow{\sim} \mathbf{J}_\mathfrak{o}$  are compatible with the action of  $\mathbf{e}^s$ .

Let  $\text{Irr}_s(\mathbf{H}_\mathfrak{o}^1)$  be the set of all  $E \in \text{Irr}(\mathbf{H}_\mathfrak{o}^1)$  with the following property: there exists a linear isomorphism  $\mathbf{e}_s : E \rightarrow E$  such that for any  $w \cdot \lambda \in I_\mathfrak{o}$  and any  $e \in E$  we have

$$\mathbf{e}_s((T_w 1_\lambda)(e)) = (T_{\mathbf{e}^s(w)} 1_{\mathbf{e}^s(\lambda)})(\mathbf{e}_s(e)).$$

(Such  $\mathbf{e}_s$  is clearly unique up to a nonzero scalar, if it exists.) We assume that for any  $E \in \text{Irr}_s(\mathbf{H}_\mathfrak{o}^1)$ , an  $\mathbf{e}_s$  as above has been chosen; we can assume that  $\mathbf{e}_s$  has finite order (since  $\mathbf{e}^s : I_\mathfrak{o} \rightarrow I_\mathfrak{o}$  has finite order); moreover, when  $s = 0$  we have  $\text{Irr}_s(\mathbf{H}_\mathfrak{o}^1) = \text{Irr}(\mathbf{H}_\mathfrak{o}^1)$  and for any  $E$  in this set we can take  $\mathbf{e}_s = 1$ . If  $E \in \text{Irr}(H_\mathfrak{o}^1)$  we can view  $E$  as a simple  $\mathbf{J}_\mathfrak{o}$ -module via  $\psi_\mathfrak{o}^1$ ; we denote this  $\mathbf{J}_\mathfrak{o}$ -module by  $E^\infty$ . Moreover we can view  $\bar{\mathbf{Q}}_l(v) \otimes E^\infty$  as a simple  $\mathbf{H}_\mathfrak{o}^v$ -module via  $\psi_\mathfrak{o}^v$ ; we denote this  $\mathbf{H}_\mathfrak{o}^v$ -module by  $E^v$ . If in addition we have  $E \in \text{Irr}_s(H_\mathfrak{o}^1)$ , then  $\mathbf{e}_s$  can be viewed as a  $\bar{\mathbf{Q}}_l$ -linear isomorphism  $E^\infty \rightarrow E^\infty$  (denoted again by  $\mathbf{e}_s$ ) and as a  $\bar{\mathbf{Q}}_l(v)$ -linear isomorphism  $E^v \rightarrow E^v$  (denoted again by  $\mathbf{e}_s$ ).

Note that for any  $\xi \in \mathbf{J}_\mathfrak{o}$ ,  $e \in E^\infty$  we have  $\mathbf{e}_s(\xi(e)) = \mathbf{e}^s(\xi)(\mathbf{e}_s(e))$ ; for any  $\xi' \in \mathbf{H}_\mathfrak{o}$ ,  $e' \in E^v$  we have  $\mathbf{e}_s(\xi'(e')) = \mathbf{e}^s(\xi')(\mathbf{e}_s(e'))$ .

**2.5.** For  $s \in \mathbf{Z}$  let

$$I^s = \{w \cdot \lambda \in I; w(\lambda) = \mathbf{e}^{-s}(\lambda)\}.$$

For any two-sided cell  $\mathbf{c}$  of  $I$  we set

$$\mathbf{c}^s = I^s \cap \mathbf{c}.$$

We show:

- (a) If  $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$  and  $i \in \mathbf{c}$ ,  $j \in I$  satisfy  $t_i! t_j t_{\mathbf{e}^s(i)} \neq 0$ , then  $j \in \mathbf{c}^s$ .
- (b) If  $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$ , then  $\mathbf{c}^s \neq \emptyset$ .

We prove (a). Let  $i = w \cdot \lambda$ ,  $j = z \cdot \lambda'$ . From our assumption we have  $t_{z \cdot \lambda'} t_{\mathbf{e}^s(w) \cdot \mathbf{e}^s(\lambda)} \neq 0$  (which implies  $\lambda' = \mathbf{e}^s(w(\lambda))$ ) and  $t_{w^{-1} \cdot w(\lambda)} t_{z \cdot \lambda'} \neq 0$  (which implies  $w(\lambda) = z(\lambda')$ ). We deduce that  $z(\lambda') = \mathbf{e}^{-s}(\lambda')$  so that  $j \in I^s$ . Since  $t_i! t_j \neq 0$  and  $i! \in \mathbf{c}$  we must have  $j \in \mathbf{c}$ . Thus we have  $j \in I^s \cap \mathbf{c}$  and (a) is proved.

We prove (b). Let  $i \in \mathbf{c}$ . By assumption we have  $\mathbf{e}^s(i) \in \mathbf{c}$ ; by Q10 in 1.9 we have  $i^! \in \mathbf{c}$ . Using 1.15(d) with  $i, i'$  replaced by  $i^!, \mathbf{e}^s(i)$  we see that for some  $j = z \cdot \lambda' \in I$  we have  $t_{i^!} t_j t_{\mathbf{e}^s(i)} \neq 0$ . Using (a) we deduce that  $j \in \mathbf{c}^s$  and (b) is proved.

### 3. SHEAVES ON $\tilde{\mathcal{B}}^2$

**3.1.** Let  $\tilde{\mathcal{B}} = G/\mathbf{U}$ . We have  $\tilde{\mathcal{B}}^2 = \sqcup_{w \in W} \tilde{\mathcal{O}}_w$  where

$$\tilde{\mathcal{O}}_w = \{(x\mathbf{U}, y\mathbf{U}) \in \tilde{\mathcal{B}}^2; x^{-1}y \in G_w\}.$$

The closure of  $\tilde{\mathcal{O}}_w$  in  $\tilde{\mathcal{B}}^2$  is  $\tilde{\mathcal{O}}_w = \cup_{y \in W; y \leq w} \tilde{\mathcal{O}}_y$ . For  $w \in W$  and  $\omega \in \kappa_0^{-1}(w)$  we define  $G_w \rightarrow \mathbf{T}$  by  $g \mapsto g_\omega$  where  $g \in \mathbf{U}\omega g_\omega \mathbf{U}$ ,  $g_\omega \in \mathbf{T}$ . We define  $j^\omega : \tilde{\mathcal{O}}_w \rightarrow \mathbf{T}$  by  $j^\omega(x\mathbf{U}, y\mathbf{U}) = (x^{-1}y)_\omega$ . For  $\lambda \in \mathfrak{s}_\infty$  we set  $L_\lambda^\omega = (j^\omega)^* L_\lambda$ , a local system on  $\tilde{\mathcal{O}}_w$ . Let  $L_\lambda^{\omega\sharp}$  be its extension to an intersection cohomology complex on  $\tilde{\mathcal{O}}_w$  viewed as a complex on  $\tilde{\mathcal{B}}^2$ , equal to 0 on  $\tilde{\mathcal{B}}^2 - \tilde{\mathcal{O}}_w$ . We shall view  $L_\lambda^\omega$  as a constructible sheaf on  $\tilde{\mathcal{B}}^2$  which is 0 on  $\tilde{\mathcal{B}}^2 - \tilde{\mathcal{O}}_w$ . Let  $\mathbf{L}_\lambda^\omega = L_\lambda^{\omega\sharp} \langle |w| + \nu + 2\rho \rangle$ , a simple perverse sheaf on  $\tilde{\mathcal{B}}^2$ .

(a) *In the remainder of this section we fix a two-sided cell  $\mathbf{c}$  of  $I$  and we set  $a = a(i)$  for some/any  $i \in \mathbf{c}$ . We define  $\mathfrak{o} \in W \setminus \mathfrak{s}_\infty$  by  $\mathbf{c} \subset I_{\mathfrak{o}}$ . We denote by  $n$  the smallest integer in  $\mathbf{N}^*$  such that  $\mathfrak{o} \subset \mathfrak{s}_n$ . We shall assume that  $\Psi$  in 2.3 acts as 1 on the finite subset  $\{t \in \mathbf{T}; t^n \in \mathbf{T} \cap N_0 \mathbf{T}\}$  of  $\mathbf{T}$ .*

In particular,  $\Psi(t) = t$  for any  $t \in \mathbf{T}_n$  (hence  $\Psi(\lambda) = \lambda$  for any  $\lambda \in \mathfrak{s}_n$ ).

Now, if  $w \in W, \omega \in \kappa_0^{-1}(w), \lambda \in \mathfrak{s}_n$ , then  $L_\lambda^\omega|_{\tilde{\mathcal{O}}_w}$ ,  $L_\lambda^{\omega\sharp}$  and  $\mathbf{L}_\lambda^\omega$  can be regarded naturally as objects in the mixed derived category of pure weight zero. Moreover,  $L_\lambda^\omega|_{\tilde{\mathcal{O}}_w}$  (resp.  $L_\lambda^{\omega\sharp}, \mathbf{L}_\lambda^\omega$ ) is (noncanonically) isomorphic to  $L_\lambda^{\dot{\omega}}|_{\tilde{\mathcal{O}}_w}$  (resp.  $L_\lambda^{\dot{\omega}\sharp}, \mathbf{L}_\lambda^{\dot{\omega}}$ ) in the mixed derived category. (It is enough to show that if  $t, t' \in \mathbf{T}, t^n = t' = \dot{w}\omega^{-1}$  and  $h_{t'} : \mathbf{T} \rightarrow \mathbf{T}$  is translation by  $t'$ , then  $t$  defines an isomorphism  $h_{t'}^* L_\lambda \rightarrow L_\lambda$ ; see [L16, 1.5].)

We define  $\tilde{\mathfrak{h}} : \tilde{\mathcal{B}}^2 \rightarrow \tilde{\mathcal{B}}^2$  by  $(x\mathbf{U}, y\mathbf{U}) \mapsto (y\mathbf{U}, x\mathbf{U})$ .

We define an action of  $G \times \mathbf{T}^2$  on  $\tilde{\mathcal{B}}^2$  (resp. on  $\mathbf{T}$ ) by

$$(g, t_1, t_2) : (x\mathbf{U}, y\mathbf{U}) \mapsto (gxt_1^n \mathbf{U}, gyt_2^n \mathbf{U})$$

(resp. by  $(g, t_1, t_2) : t \mapsto w^{-1}(t_1)^{-n} t t_2^n$ ). For any  $w \in W$ , the  $G \times \mathbf{T}^2$ -action leaves stable  $\tilde{\mathcal{O}}_w$  and its restriction to  $\tilde{\mathcal{O}}_w$  is transitive; moreover,  $j^\omega$  is compatible with actions of  $G \times \mathbf{T}^2$  on  $\tilde{\mathcal{O}}_w$  and  $\mathbf{T}$ .

If  $\lambda \in \mathfrak{s}_n$  then  $L_\lambda$  is a  $G \times \mathbf{T}^2$ -equivariant local system on  $\mathbf{T}$  hence  $L_w^\lambda$  is a  $G \times \mathbf{T}^2$ -equivariant local system on  $\tilde{\mathcal{O}}_w$ . By [L16, 2.1], the following holds.

(c) *For fixed  $w \in W, \omega \in \kappa_0^{-1}(w)$ , the local systems  $L_\lambda^\omega$  with  $\lambda \in \mathfrak{s}_n$  form a set of representatives for the isomorphism classes of irreducible  $G \times \mathbf{T}^2$ -equivariant local systems on  $\tilde{\mathcal{O}}_w$ .*

**3.2.** We define  $p_{01} : \tilde{\mathcal{B}}^3 \rightarrow \tilde{\mathcal{B}}^2$ ,  $p_{12} : \tilde{\mathcal{B}}^3 \rightarrow \tilde{\mathcal{B}}^2$ ,  $p_{02} : \tilde{\mathcal{B}}^3 \rightarrow \tilde{\mathcal{B}}^2$  by

$$\begin{aligned} p_{01}(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) &= (x\mathbf{U}, y\mathbf{U}), p_{12}(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) = (y\mathbf{U}, z\mathbf{U}), \\ p_{02}(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) &= (x\mathbf{U}, z\mathbf{U}). \end{aligned}$$

For any  $L \in \mathcal{D}(\tilde{\mathcal{B}}^2)$ ,  $L' \in \mathcal{D}(\tilde{\mathcal{B}}^2)$ , we set

$$L \circ L' = p_{02!}(p_{01}^* L \otimes p_{12}^* L') \in \mathcal{D}(\tilde{\mathcal{B}}^2).$$

This defines a monoidal structure on  $\mathcal{D}(\tilde{\mathcal{B}}^2)$ . Thus, if  ${}^i L \in \mathcal{D}(\tilde{\mathcal{B}})$  for  $i = 1, \dots, k$ , then  ${}^1 L \circ {}^2 L \circ \dots \circ {}^k L \in \mathcal{D}(\tilde{\mathcal{B}})$  is well defined. Note that, if  $L \in \mathcal{D}_m(\tilde{\mathcal{B}}^2)$ ,  $L' \in \mathcal{D}_m(\tilde{\mathcal{B}}^2)$  then  $L \circ L'$  is naturally in  $\mathcal{D}_m(\tilde{\mathcal{B}}^2)$ .

**3.3.** Now assume that  $w, w' \in W$ ,  $\omega \in \kappa_0^{-1}(w)$ ,  $\omega' \in \kappa_0^{-1}(w')$ ,  $\lambda, \lambda' \in \mathfrak{s}_\infty$ . From [L16, 2.3] we see that:

(a) if  $w'(\lambda') \neq \lambda$ , then  $L_\lambda^\omega \circ L_{\lambda'}^{\omega'} = 0$ .

**3.4.** Now assume that  $w, w' \in W$ ,  $\omega \in \kappa_0^{-1}(w)$ ,  $\omega' \in \kappa_0^{-1}(w')$ ,  $\lambda, \lambda' \in \mathfrak{s}_\infty$ . Let  $\Xi$  be the set of all  $(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{B}}^3$  such that  $x^{-1}y \in \mathbf{U}\omega t\mathbf{U}$ ,  $y^{-1}z \in \mathbf{U}\omega' t'\mathbf{U}$  for some  $t, t'$  in  $\mathbf{T}$  (which are in fact uniquely determined). Define  $c : \Xi \rightarrow \mathbf{T} \times \mathbf{T}$  by  $c(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) = (t, t')$  where  $x^{-1}y \in \mathbf{U}\omega t\mathbf{U}$ ,  $y^{-1}z \in \mathbf{U}\omega' t'\mathbf{U}$ . Define  $p'_{02} : \Xi \rightarrow \tilde{\mathcal{B}}^2$  by  $(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \mapsto (x\mathbf{U}, z\mathbf{U})$ . From the definitions we see that

$$(a) \quad L_\lambda^\omega \circ L_{\lambda'}^{\omega'} = p'_{02!}(c^*(L_\lambda \boxtimes L_{\lambda'})).$$

We show:

(b) If  $w'(\lambda') = \lambda$  and  $|ww'| = |w| + |w'|$ , then we have canonically  $L_\lambda^\omega \circ L_{\lambda'}^{\omega'} = L_{\lambda'}^{\omega\omega'} \otimes \mathfrak{L}$ , with  $\mathfrak{L}$  as in 0.2.

Let  $Y = \{(x\mathbf{U}, z\mathbf{U}, t, t') \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times \mathbf{T}; x^{-1}z \in \mathbf{U}\omega t\mathbf{U}\omega' t'\mathbf{U}\}$ . We define  $\Xi \rightarrow Y$  by  $(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \mapsto (x\mathbf{U}, z\mathbf{U}, t, t')$  where  $t, t'$  in  $\mathbf{T}$  are given by  $x^{-1}y \in \mathbf{U}\omega t\mathbf{U}$ ,  $y^{-1}z \in \mathbf{U}\omega' t'\mathbf{U}$ . This is an isomorphism since  $|ww'| = |w| + |w'|$ . We identify  $\Xi = Y$  through this isomorphism. Then  $c : \Xi \rightarrow \mathbf{T} \times \mathbf{T}$  becomes  $c : Y \rightarrow \mathbf{T} \times \mathbf{T}$ ,  $(x\mathbf{U}, z\mathbf{U}, t, t') \mapsto (t, t')$ . We define  $h : \mathbf{T} \times \mathbf{T} \rightarrow \mathbf{T}$  by  $h(t, t') = w'^{-1}(t)t'$ . We have

$$Y = \{(x\mathbf{U}, z\mathbf{U}, t, t') \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times \mathbf{T}; x^{-1}z \in \mathbf{U}\omega\omega' h(t, t')\mathbf{U}\}.$$

Define  $j : Y \rightarrow \tilde{\mathcal{O}}_{ww'}$  by  $(x\mathbf{U}, z\mathbf{U}, t, t') \mapsto (x\mathbf{U}, z\mathbf{U})$ . Let  $j' = j^{\omega\omega'} : \tilde{\mathcal{O}}_{ww'} \rightarrow \mathbf{T}$ . Using (a) and the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{c} & \mathbf{T} \times \mathbf{T} \\ j \downarrow & & h \downarrow \\ \tilde{\mathcal{O}}_{ww'} & \xrightarrow{j'} & \mathbf{T} \end{array}$$

we see that

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega'} = j_! c^*(L_\lambda \boxtimes L_{\lambda'}) = j'^* h_!(L_\lambda \boxtimes L_{\lambda'})$$

. Since  $L_{\lambda'}^{\omega\omega'} \otimes \mathfrak{L} = j'^*(L_{\lambda'} \otimes \mathfrak{L})$ , we see that to prove (b) it is enough to show that  $h_!(L_\lambda \boxtimes L_{\lambda'}) = L_{\lambda'} \otimes \mathfrak{L}$  (assuming that  $w'(\lambda') = \lambda$ ). This is proved as in the last paragraph of [L16, 2.4].



**3.5.** Let  $\sigma \in S$  and let  $\omega \in \kappa_0^{-1}(\sigma)$ ,  $\lambda' \in \mathfrak{s}_\infty$ . Define  $\delta_\omega : \mathbf{U}_\sigma - \{1\} \rightarrow \mathbf{T}$  by  $\xi \mapsto t_\xi^{-1}$  where  $t_\xi \in \mathbf{T}$  is given by  $\omega^{-1}\xi^{-1}\omega \in \mathbf{U}\omega^{-1}t_\xi\mathbf{U}$ ; let  $\mathcal{E} = \delta_\omega^* L_{\lambda'}^*$ . Let  $\delta' : \mathbf{U}_\sigma - \{1\} \rightarrow \mathbf{p}$  be the obvious map. From the definitions we see that:

(a)  $\delta'_! \mathcal{E} = 0$  if  $\sigma \notin W_{\lambda'}$ ;  $\delta'_! \mathcal{E} \simeq \{\bar{\mathbf{Q}}_l \langle -2 \rangle, \bar{\mathbf{Q}}_l[-1]\}$  if  $\sigma \in W_{\lambda'}$ .

Consider the diagram  $\mathbf{T} \xleftarrow{\tilde{k}} \mathbf{T} \times (\mathbf{U}_\sigma - \{1\}) \xrightarrow{\tilde{h}} \mathbf{T}$  where  $\tilde{k} : (t, \xi) \mapsto t_\xi^{-1}$  and  $\tilde{h} : (t, \xi) \mapsto tt_\xi^{-1}$ . We show:

(b) Let  $\lambda' \in \mathfrak{s}_\infty$ . If  $\sigma \notin W_{\lambda'}$ , then  $\tilde{h}_! \tilde{k}^* L_{\lambda'} = 0$ . If  $\sigma \in W_{\lambda'}$  then  $\tilde{h}_! \tilde{k}^* L_{\lambda'}^* \simeq \{\bar{\mathbf{Q}}_l \langle -2 \rangle, \bar{\mathbf{Q}}_l[-1]\}$ .

We have  $\tilde{k}^* L_{\lambda'}^* = \bar{\mathbf{Q}}_l \boxtimes \mathcal{E}$ . Now  $\tilde{h} = \tilde{h}' y$  where  $y : \mathbf{T} \times (\mathbf{U}_\sigma - \{1\}) \rightarrow \mathbf{T} \times (\mathbf{U}_\sigma - \{1\})$  is  $(t, \xi) \mapsto (tt_\xi^{-1}, \xi)$  and  $\tilde{h}' : \mathbf{T} \times (\mathbf{U}_\sigma - \{1\}) \rightarrow \mathbf{T}$  is  $(t, \xi) \mapsto t$ . Clearly,  $y_!(\bar{\mathbf{Q}}_l \boxtimes \mathcal{E}) = \bar{\mathbf{Q}}_l \boxtimes \mathcal{E}$ . It remains to note that  $\tilde{h}'_!(\bar{\mathbf{Q}}_l \boxtimes \mathcal{E})$  is 0 if  $\sigma \notin W_{\lambda'}$  and is  $\simeq \{\bar{\mathbf{Q}}_l \langle -2 \rangle, \bar{\mathbf{Q}}_l[-1]\}$  if  $\sigma \in W_{\lambda'}$ . (This follows from (a).)

We show:

(c) Assume that  $\lambda \in \mathfrak{s}_\infty$  satisfies  $\sigma \in W_\lambda$  and that  $\omega \in \{\dot{\sigma}, \dot{\sigma}^{-1}\}$ . Then we have canonically  $L_\lambda^\omega = L_\lambda^{\omega^{-1}}$ .

Define  $\zeta : \mathbf{T} \rightarrow \mathbf{T}$  by  $t \mapsto \omega^2 t$ . It is enough to show that  $\zeta^* L_\lambda = L_\lambda$  canonically. For  $t \in \mathbf{T}$  we have  $(\zeta^* L_\lambda)_t = (L_\lambda)_{\omega^2 t} = (L_\lambda)_{\alpha_\sigma(-1)} \otimes (L_\lambda)_t$ . Hence it is enough to show that we have canonically  $(L_\lambda)_{\alpha_\sigma(-1)} = \bar{\mathbf{Q}}_l$ . It is also enough to show that  $\alpha_\sigma^* L_\lambda = \bar{\mathbf{Q}}_l$ . This follows from  $\alpha_\sigma \in R_\lambda$ .

**3.6.** Now assume that  $w = w' = \sigma \in S$ ,  $\omega \in \kappa_0^{-1}(\sigma)$ ,  $\lambda, \lambda' \in \mathfrak{s}_\infty$  are such that  $\sigma(\lambda') = \lambda$ . In this subsection we show:

(a) If  $\sigma \notin W_\lambda$ , then  $L_\lambda^\omega \circ L_{\lambda'}^{\omega^{-1}} = L_{\lambda'}^1 \langle -2 \rangle \otimes \mathfrak{L}$ .

(b) If  $\sigma \in W_\lambda$ , then

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega^{-1}} \simeq \{L_{\lambda'}^1 \langle -2 \rangle \otimes \mathfrak{L}, L_{\lambda'}^\omega \langle -2 \rangle \otimes \mathfrak{L}, L_{\lambda'}^\omega[-1] \otimes \mathfrak{L}\}.$$

(Note that the conditions  $\sigma \in W_\lambda$  and  $\sigma \in W_{\lambda'}$  are equivalent.) With the notation of 3.4, we have

$$\Xi = \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{B}}^3; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}z \in \mathbf{U}\omega^{-1}t'\mathbf{U} \text{ for some } t, t' \text{ in } \mathbf{T}\}.$$

If  $(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \Xi$  then  $x^{-1}z \in \mathbf{U}\omega\mathbf{U}\omega^{-1}w'^{-1}(t)t'\mathbf{U}$ ; in particular we have  $x^{-1}z \in \mathbf{B} \cup \mathbf{B}\omega\mathbf{B}$ . Thus,  $\Xi$  can be partitioned as  $\tilde{\mathcal{B}}^I \cup \tilde{\mathcal{B}}^{II}$  where

$$\tilde{\mathcal{B}}^I = \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \Xi; x^{-1}z \in \mathbf{B}\}$$

is a closed subset and

$$\tilde{\mathcal{B}}^{II} = \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \Xi; x^{-1}z \in \mathbf{B}\omega\mathbf{B}\}$$

is an open subset. The map  $p'_{02} : \Xi \rightarrow \tilde{\mathcal{B}}^2$  (see 3.4) restricts to maps

$$p_{02}^I : \tilde{\mathcal{B}}^I \rightarrow \tilde{\mathcal{O}}_1, p_{02}^{II} : \tilde{\mathcal{B}}^{II} \rightarrow \tilde{\mathcal{O}}_\sigma;$$

using 3.4(a) we deduce

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega^{-1}} \simeq \{p_{02}^I(c^*(L_\lambda \boxtimes L_{\lambda'})), \quad p_{02}^{II}(c^*(L_\lambda \boxtimes L_{\lambda'}))\}.$$

We show:

$$(c) \quad p_{02}^I(c^*(L_\lambda \boxtimes L_{\lambda'})) = L_{\lambda'}^1 \otimes \mathfrak{L}\langle -2 \rangle.$$

We have

$$\begin{aligned} \tilde{\mathcal{B}}^I &= \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{B}}^3; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}z \in \mathbf{U}\omega^{-1}t'\mathbf{U} \\ &\text{for some } t, t' \text{ in } \mathbf{T}, x^{-1}z \in \mathbf{B}\}, \end{aligned}$$

or equivalently

$$\tilde{\mathcal{B}}^I = \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{B}}^3; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, x^{-1}z \in \mathbf{U}\sigma(t)t'\mathbf{U} \text{ for some } t, t' \text{ in } \mathbf{T}\}.$$

Let  $Y = \{(x\mathbf{U}, z\mathbf{U}, t, t') \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times \mathbf{T}; x^{-1}z \in \mathbf{U}\sigma(t)t'\mathbf{U}\}$ . We define  $d : \tilde{\mathcal{B}}^I \rightarrow Y$  by  $(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \mapsto (x\mathbf{U}, z\mathbf{U}, t, t')$  where  $t, t'$  in  $\mathbf{T}$  are as in the last formula for  $\tilde{\mathcal{B}}^I$ . The fibre of  $d$  at  $(x\mathbf{U}, z\mathbf{U}, t, t') \in Y$  can be identified with  $\{y\mathbf{U}; y \in x\mathbf{U}\omega t\mathbf{U}\}$ , an affine line. Thus,  $d$  is an affine line bundle. We have a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{c^I} & \mathbf{T} \times \mathbf{T} \\ j^I \downarrow & & h \downarrow \\ \tilde{\mathcal{O}}_1 & \xrightarrow{\tilde{j}^I} & \mathbf{T} \end{array}$$

where  $c^I : Y \rightarrow \mathbf{T} \times \mathbf{T}$  is  $(x\mathbf{U}, z\mathbf{U}, t, t') \mapsto (t, t')$ ,  $j^I : Y \rightarrow \tilde{\mathcal{O}}_1$  is  $(x\mathbf{U}, z\mathbf{U}, t, t') \mapsto (x\mathbf{U}, z\mathbf{U})$ ,  $\tilde{j}^I = j^1 : \tilde{\mathcal{O}}_1 \rightarrow \mathbf{T}$ ,  $h : \mathbf{T} \times \mathbf{T} \rightarrow \mathbf{T}$  is  $(t, t') \mapsto \sigma(t)t'$ . As in 3.4 we have  $h_!(L_\lambda \boxtimes L_{\lambda'}) = L_{\lambda'} \otimes \mathfrak{L}$  (since  $\sigma(\lambda') = \lambda$ ). It follows that

$$(j^I)!(c^I)^*(L_\lambda \boxtimes L_{\lambda'}) = (\tilde{j}^I)^*h_!(L_\lambda \boxtimes L_{\lambda'}) = (\tilde{j}^I)^*L_{\lambda'} \otimes \mathfrak{L}.$$

Hence

$$\begin{aligned} p_{02}^I(c^*(L_\lambda \boxtimes L_{\lambda'})) &= (j^I)!d_!d^*(c^I)^*(L_\lambda \boxtimes L_{\lambda'}) = (j^I)!(c^I)^*(L_\lambda \boxtimes L_{\lambda'})\langle -2 \rangle \\ &= (\tilde{j}^I)^*L_{\lambda'} \otimes \mathfrak{L}\langle -2 \rangle = L_{\lambda'}^1 \otimes \mathfrak{L}\langle -2 \rangle. \end{aligned}$$

This proves (c). Next we show that

(d)  $p_{02}^{II}(c^*(L_\lambda \boxtimes L_{\lambda'}))$  is 0 if  $\sigma \notin W_{\lambda'}$  and is  $\simeq \{L_{\lambda'}^\omega \langle -2 \rangle, L_{\lambda'}^\omega[-1]\}$  if  $\sigma \in W_{\lambda'}$ .

We have

$$\begin{aligned} \tilde{\mathcal{B}}^{II} &= \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{B}}^3; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}z \in \mathbf{U}\omega^{-1}t'\mathbf{U} \\ &\text{for some } t, t' \text{ in } \mathbf{T}, x^{-1}z \in \mathbf{U}\omega t_1\mathbf{U} \text{ for some } t_1 \in \mathbf{T}\}. \end{aligned}$$

Let  $(x\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{O}}_\sigma$ . We can write uniquely  $z = x\xi_0\omega t_1 u_1$  where  $\xi_0 \in \mathbf{U}_\sigma$ ,  $t_1 \in \mathbf{T}$ ,  $u_1 \in \mathbf{U}$ . The fibre  $\Phi$  of  $p_{02}^{II}$  at  $(x\mathbf{U}, z\mathbf{U})$  can be identified with

$$\begin{aligned} &\{y\mathbf{U} \in G/U\mathbf{U}; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}z \in \mathbf{U}\omega^{-1}t'\mathbf{U}\} \\ &= \{y\mathbf{U} \in G/U\mathbf{U}; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}x\xi_0\omega t_1 u_1 \in \mathbf{U}\omega^{-1}t'\mathbf{U}\}. \end{aligned}$$

Setting  $x^{-1}y = \xi\omega t u'$  where  $\xi \in \mathbf{U}_\sigma$ , we can identify

$$\begin{aligned} \Phi &= \{(t, t', \xi) \in \mathbf{T} \times \mathbf{T} \times \mathbf{U}_\sigma; u'^{-1}t^{-1}\omega^{-1}\xi^{-1}\xi_0\omega t_1 \in \mathbf{U}\omega^{-1}t'\mathbf{U}\} \\ &= \{(t, t', \xi) \in \mathbf{T} \times \mathbf{T} \times \mathbf{U}_\sigma; \omega^{-1}\xi^{-1}\xi_0\omega \in \mathbf{U}\omega^{-1}\sigma(t)t't_1^{-1}\mathbf{U}\} \\ &= \{(t, t', \xi) \in \mathbf{T} \times \mathbf{T} \times (\mathbf{U}_\sigma - \{\xi_0\}); t_{\xi^{-1}\xi_0} = \sigma(t)t't_1^{-1}\} \end{aligned}$$

where for  $\xi_1 \in \mathbf{U}_\sigma - \{1\}$  we define  $t_{\xi_1} \in \mathbf{T}$  by  $\omega^{-1}\xi_1^{-1}\omega \in \mathbf{U}\omega^{-1}t_{\xi_1}\mathbf{U}$ . Let

$$Y' = \{(x\mathbf{U}, z\mathbf{U}, t, t', \xi_1) \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times \mathbf{T} \times (\mathbf{U}_\sigma - \{1\}); x^{-1}z \in \mathbf{U}_\sigma\omega\sigma(t)t't_{\xi_1}^{-1}\mathbf{U}\},$$

$$Y'_1 = \{(x\mathbf{U}, z\mathbf{U}, t'_1, \xi_1) \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times (\mathbf{U}_\sigma - \{1\}); x^{-1}z \in \mathbf{U}_\sigma\omega t'_1 t_{\xi_1}^{-1}\mathbf{U}\}.$$

We see that  $\tilde{\mathcal{B}}^{II}$  may be identified with  $Y'$ . (The identification is via

$$(x\mathbf{U}, z\mathbf{U}, t, t', \xi_1) \mapsto (x\mathbf{U}, x\xi_0\xi_1^{-1}\omega t\mathbf{U}, z\mathbf{U})$$

where  $\xi_0 \in \mathbf{U}_\sigma$  is given by  $x^{-1}z \in \xi_0\omega\mathbf{T}\mathbf{U}$ .) Under this identification,  $p_{02}^{II}$  becomes the composition  $fj^{II}$  where  $j^{II} : Y' \rightarrow Y'_1$  is

$$(x\mathbf{U}, z\mathbf{U}, t, t', \xi_1) \mapsto (x\mathbf{U}, z\mathbf{U}, s(t)t', \xi_1)$$

and  $f : Y'_1 \rightarrow \tilde{\mathcal{O}}_\sigma$  is

$$(x\mathbf{U}, z\mathbf{U}, t'_1, \xi_1) \mapsto (x\mathbf{U}, z\mathbf{U});$$

moreover, the local system  $c^*(L_\lambda \boxtimes L_{\lambda'})$  on  $\tilde{\mathcal{B}}^{II}$  becomes the local system  $(c^{II})^*(L_\lambda \boxtimes L_{\lambda'})$  on  $Y'$  where  $c^{II} : Y' \rightarrow \mathbf{T} \times \mathbf{T}$  is  $(x\mathbf{U}, z\mathbf{U}, t, t', \xi_1) \mapsto (t, t')$ . We have a diagram with cartesian squares

$$\begin{array}{ccccc} Y' & \xrightarrow{c^{II}} & \mathbf{T} \times \mathbf{T} & & \\ & \downarrow j^{II} & & \downarrow h & \\ \mathbf{T} \times (\mathbf{U}_\sigma - \{1\}) & \xleftarrow{\tilde{j}'} & Y'_1 & \xrightarrow{\tilde{j}^{II}} & \mathbf{T} \\ & \downarrow \tilde{h} & & \downarrow f & \\ & \mathbf{T} & \xleftarrow{j'} & \tilde{\mathcal{O}}_s & \end{array}$$

where  $\tilde{j}^{II} : Y'_1 \rightarrow \mathbf{T}$  is  $(x\mathbf{U}, z\mathbf{U}, t'_1, \xi_1) \mapsto t'_1$ ,  $j' : \tilde{\mathcal{O}}_\sigma \rightarrow \mathbf{T}$  is  $j^\omega$ ,  $\tilde{j}' : Y'_1 \rightarrow \mathbf{T} \times (\mathbf{U}_\sigma - \{1\})$  is  $(x\mathbf{U}, z\mathbf{U}, t'_1, \xi_1) \mapsto (t'_1, \xi_1)$ ,  $h : \mathbf{T} \times \mathbf{T} \rightarrow \mathbf{T}$  is  $(t, t') \mapsto \sigma(t)t'$  and  $\tilde{h}'$  is as in 3.5.

Let  $L' = (\tilde{j}^{II})^* L_{\lambda'}$  (a local system on  $Y'_1$ ). Let  $L'' = j'^* L_{\lambda'} = L_{\lambda'}^\omega$  (a local system on  $\tilde{\mathcal{O}}_\sigma$ ). Define  $\tilde{f} : Y'_1 \rightarrow \mathbf{T}$  by  $(x\mathbf{U}, z\mathbf{U}, t'_1, \xi_1) \mapsto t_{\xi_1}^{-1}$ . Let  $\tilde{L} = \tilde{f}^* L_{\lambda'}$  (a local system on  $Y'_1$ ). The stalk of  $L'$  at  $(x\mathbf{U}, z\mathbf{U}, t'_1, \xi_1) \in Y'_1$  is  $(L_{\lambda'})_{t'_1}$ . The stalk of  $f^* L''$  at  $(x\mathbf{U}, z\mathbf{U}, t'_1, \xi_1) \in Y'_1$  is  $(L_{\lambda'})_{t'_1 t_{\xi_1}^{-1}} = (L_{\lambda'})_{t'_1} \otimes (L_{\lambda'})_{t_{\xi_1}^{-1}}$ . Thus we have  $L' = f^* L'' \otimes \tilde{L}^*$ .

As in 3.4 we have  $h_!(L_\lambda \boxtimes L_{\lambda'}) = L_{\lambda'} \otimes \mathfrak{L}$  (since  $\sigma(\lambda') = \lambda$ ). Using the cartesian diagrams above, we see that

$$\begin{aligned} p_{02!}^{II}(c^*(L_\lambda \boxtimes L_{\lambda'})) &= f_! j_!^{II}(c^{II})^*(L_\lambda \boxtimes L_{\lambda'}) \\ &= f_! j_!^{II}(c^{II})^*(L_\lambda \boxtimes L_{\lambda'}) = f_!(\tilde{j}^{II})^* h_!(L_\lambda \boxtimes L_{\lambda'}) = f_!(\tilde{j}^{II})^* L_{\lambda'} \otimes \mathfrak{L} \\ &= f_!(L') \otimes \mathfrak{L} = f_!(f^* L'' \otimes \tilde{L}^*) \otimes \mathfrak{L} = L'' \otimes f_!(\tilde{L}^*) \otimes \mathfrak{L} = L'' \otimes f_! \tilde{j}'^* \tilde{k}^*(L_{\lambda'}^*) \\ &= L'' \otimes f_! \tilde{j}'^* \tilde{k}^*(L_{\lambda'}^*) = L'' \otimes j'^* \tilde{h}_! \tilde{k}^*(L_{\lambda'}^*) = L'' \otimes j'^* \tilde{h}_! \tilde{k}^*(L_{\lambda'}^*). \end{aligned}$$

Here  $\tilde{k}$  is as in 3.5. Using 3.5(b) we see that this is 0 if  $\sigma \notin W_{\lambda'}$  and is  $\simeq \{L''\langle -2 \rangle, L''[-1]\}$  if  $\sigma \in W_{\lambda'}$ . This proves (d). Now (a),(b) follow from (c),(d).

**3.7.** Now assume that  $w \in W$ ,  $\sigma \in S$ ,  $\omega \in \{\dot{\sigma}, \dot{\sigma}^{-1}\}$ ,  $\omega' \in \kappa_0^{-1}(w)$ ,  $\lambda, \lambda' \in \mathfrak{s}_\infty$  are such that  $w(\lambda') = \lambda$ ,  $|\sigma w| < |w|$ . We show:

- (a) If  $\sigma \notin W_\lambda$  then  $L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} = L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}$ .
- (b) If  $\sigma \in W_\lambda$ , then

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} \simeq \{L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}, L_{\lambda'}^{\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}, L_{\lambda'}^{\omega'} [-1] \otimes \mathfrak{L} \otimes \mathfrak{L}\}.$$

Using 3.4(b), we have  $L_{\lambda'}^{\omega'} \otimes \mathfrak{L} = L_{(\sigma w)(\lambda')}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'}$ . Hence  $L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} = L_\lambda^\omega \circ L_{(\sigma w)(\lambda')}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'}$ . We now apply 3.6(a),(b) to describe  $L_\lambda^\omega \circ L_{(\sigma w)(\lambda')}^{\omega^{-1}}$ . If  $\sigma \notin W_\lambda$ , we obtain

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} = L_{(\sigma w)(\lambda')}^1 \circ L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L}.$$

By 3.4(b) this equals  $L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{l}^{\otimes 2}$ , proving (a). If  $\sigma \in W_\lambda$ , we obtain

$$\begin{aligned} L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} &\simeq \{L_{(\sigma w)\lambda'}^1 \circ L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L}, \\ &L_{(\sigma w)\lambda'}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L}, L_{(\sigma w)\lambda'}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'} [-1] \otimes \mathfrak{L}\}. \end{aligned}$$

(We have used that  $L_{(\sigma w)\lambda'}^\omega = L_{(\sigma w)\lambda'}^{\omega^{-1}}$ , see 3.5(c).) We now substitute

$$L_{(\sigma w)\lambda'}^1 \circ L_{\lambda'}^{\omega\omega'} = L_{\lambda'}^{\omega\omega'} \otimes \mathfrak{L}, L_{(\sigma w)\lambda'}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'} = L_{\lambda'}^{\omega\omega'} \otimes \mathfrak{L},$$

see 3.4(b); we obtain

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} \simeq \{L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}, L_{\lambda'}^{\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}, L_{\lambda'}^{\omega'} [-1] \otimes \mathfrak{L} \otimes \mathfrak{L}\}.$$

This proves (b).

**3.8.** Let  $\mathcal{D}^\bullet \tilde{\mathcal{B}}^2$  be the subcategory of  $\mathcal{D}(\tilde{\mathcal{B}}^2)$  consisting of objects which are restrictions of objects in the  $G \times \mathbf{T}^2$ -equivariant derived category. Let  $\mathcal{M}^\bullet \tilde{\mathcal{B}}^2$  be the subcategory of  $\mathcal{D}^\bullet \tilde{\mathcal{B}}^2$  consisting of objects which are perverse sheaves. Let  $\mathcal{M}^\preceq \tilde{\mathcal{B}}^2$  (resp.  $\mathcal{M}^\prec \tilde{\mathcal{B}}^2$ ) be the subcategory of  $\mathcal{M}^\bullet \tilde{\mathcal{B}}^2$  whose objects are perverse sheaves  $L$  such that any composition factor of  $L$  is of the form  $\mathbf{L}_\lambda^w$  for some  $w \cdot \lambda \preceq \mathbf{c}$  (resp.  $w \cdot \lambda \prec \mathbf{c}$ ). Let  $\mathcal{D}^\preceq \tilde{\mathcal{B}}^2$  (resp.  $\mathcal{D}^\prec \tilde{\mathcal{B}}^2$ ) be the subcategory of  $\mathcal{D}^\bullet \tilde{\mathcal{B}}^2$  whose objects are complexes  $L$  such that  $L^j$  is in  $\mathcal{M}^\preceq \tilde{\mathcal{B}}^2$  (resp.  $\mathcal{M}^\prec \tilde{\mathcal{B}}^2$ ) for any  $j$ . We write  $\mathcal{D}_m()$  or  $\mathcal{M}_m()$  for the mixed version of any of the categories above. Let  $\mathcal{C}^\bullet \tilde{\mathcal{B}}^2$  be the subcategory of  $\mathcal{M}^\bullet \tilde{\mathcal{B}}^2$  consisting of semisimple objects. Let  $\mathcal{C}_0^\bullet \tilde{\mathcal{B}}^2$  be the subcategory of  $\mathcal{M}_m^\bullet \tilde{\mathcal{B}}^2$  consisting of objects of pure of weight zero. Let  $\mathcal{C}^c \tilde{\mathcal{B}}^2$  be the subcategory of  $\mathcal{M}^\bullet \tilde{\mathcal{B}}^2$  consisting of objects which are direct sums of objects of the form  $\mathbf{L}_\lambda^w$  with  $w \cdot \lambda \in \mathbf{c}$ . Let  $\mathcal{C}_0^c \tilde{\mathcal{B}}^2$  be the subcategory of  $\mathcal{C}_0^\bullet \tilde{\mathcal{B}}^2$  consisting of those  $L \in \mathcal{C}_0^\bullet \tilde{\mathcal{B}}^2$  such that, as an object of  $\mathcal{C}^\bullet \tilde{\mathcal{B}}^2$ ,  $L$  belongs to  $\mathcal{C}^c \tilde{\mathcal{B}}^2$ . For  $L \in \mathcal{C}_0^\bullet \tilde{\mathcal{B}}^2$  let  $\underline{L}$  be the largest subobject of  $L$  such that as an object of  $\mathcal{C}^\bullet \tilde{\mathcal{B}}^2$ , we have  $\underline{L} \in \mathcal{C}^c \tilde{\mathcal{B}}^2$ .

**3.9.** Let  $r \geq 1$ . Let  $\mathbf{w} = (w_1, \dots, w_r) \in W^r$ ,  $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_r)$  be such that  $\omega_i \in \kappa_0^{-1}(w_i)$  for  $i = 1, \dots, r$  and  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathfrak{s}_n^r$ . We set

$$|\mathbf{w}| = |w_1| + |w_2| + \dots + |w_r|.$$

For  $J \subset [1, r]$ , let

$$\begin{aligned} \tilde{\mathcal{O}}_{\mathbf{w}}^J &= \{(x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}) \in \tilde{\mathcal{B}}^{r+1}; \\ x_{i-1}^{-1} x_i \mathbf{U} &\in \bar{G}_{w_i} \forall i \in J, x_{i-1}^{-1} x_i \in G_{w_i} \forall i \in [1, r] - J\}. \end{aligned}$$

Define  $c : \tilde{\mathcal{O}}_{\mathbf{w}}^\emptyset \rightarrow \mathbf{T}^r$  by

$$c(x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}) = ((x_0^{-1} x_1)_{w_1}, (x_1^{-1} x_2)_{w_2}, \dots, (x_{r-1}^{-1} x_r)_{w_r}).$$

Let  $M_{\boldsymbol{\lambda}}^{\mathbf{w}} \in \mathcal{D}_m(\tilde{\mathcal{B}}^{r+1})$  be the local system  $c^*(L_{\lambda_1} \boxtimes \dots \boxtimes L_{\lambda_r})$  on  $\tilde{\mathcal{O}}_{\mathbf{w}}^\emptyset$  extended by 0 on  $\tilde{\mathcal{B}}^{r+1} - \tilde{\mathcal{O}}_{\mathbf{w}}^\emptyset$ . For  $J \subset [1, r]$  we set

$$M_{\boldsymbol{\lambda}}^{\mathbf{w}, J} = p_{01}^{*-1} L \otimes p_{12}^{*-2} L \otimes \dots \otimes p_{r-1, r}^{*-r} L \in \mathcal{D}_m(\tilde{\mathcal{B}}^{r+1}),$$

$$L_{\boldsymbol{\lambda}}^{\mathbf{w}, J} = p_{0r!} M_{\boldsymbol{\lambda}}^{\mathbf{w}, J} \langle |\mathbf{w}| \rangle = {}^1 L \circ {}^2 L \circ \dots \circ {}^r L \langle |\mathbf{w}| \rangle \in \mathcal{D}_m(\tilde{\mathcal{B}}^2),$$

where  ${}^i L$  is  $L_{\lambda_i}^{\omega_i \sharp}$  for  $i \in J$  and  $L_{\lambda_i}^{\omega_i}$  for  $i \notin J$ . Note that  $M_{\boldsymbol{\lambda}}^{\mathbf{w}, \emptyset} = M_{\boldsymbol{\lambda}}^{\mathbf{w}}$ . Moreover, from [L16, 2.15] we have:

(a)  $M_{\boldsymbol{\lambda}}^{\mathbf{w}, J}$  is the intersection cohomology complex of  $\tilde{\mathcal{O}}_{\mathbf{w}}^J$  with coefficients in  $M_{\boldsymbol{\lambda}}^{\mathbf{w}}$ .

Consider the free  $\mathbf{T}^{r-1}$ -action on  $\tilde{\mathcal{B}}^{r+1}$  given by

$$\begin{aligned} (\tau_1, \tau_2, \dots, \tau_{r-1}) : (x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_{r-1} \mathbf{U}, x_r \mathbf{U}) &\mapsto \\ (x_0 \mathbf{U}, x_1 \tau_1 \mathbf{U}, \dots, x_{r-1} \tau_{r-1} \mathbf{U}, x_r \mathbf{U}). \end{aligned}$$

Note that  $\tilde{\mathcal{O}}_{\mathbf{w}}^J$  is stable under this  $\mathbf{T}^{r-1}$ -action. We also have a free  $\mathbf{T}^{r-1}$ -action on  $\mathbf{T}^r$  given by

$$(\tau_1, \tau_2, \dots, \tau_{r-1}) : (t_1, t_2, \dots, t_r) \mapsto (t_1 \tau_1, w_2^{-1}(\tau_1^{-1}) t_2 \tau_2, w_3^{-1}(\tau_2^{-1}) t_3 \tau_3, \dots, w_{r-1}^{-1}(\tau_{r-2}^{-1}) t_{r-1} \tau_{r-1}, w_r^{-1}(\tau_{r-1}^{-1}) t_r).$$

Let  $'\tilde{\mathcal{B}}^{r+1} = \mathbf{T}^{r-1} \backslash \tilde{\mathcal{B}}^{r+1}$ . Let  $'\tilde{\mathcal{O}}_{\mathbf{w}}^J = \mathbf{T}^{r-1} \backslash \tilde{\mathcal{O}}_{\mathbf{w}}^J$  (a locally closed subvariety of  $'\tilde{\mathcal{B}}^{r+1}$ ). Let  $'\mathbf{T}^r = \mathbf{T}^{r-1} \backslash \mathbf{T}^r$ . Note that  $'\tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset} = \mathbf{T}^{r-1} \backslash \tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset}$  is an open dense smooth irreducible subvariety of  $'\tilde{\mathcal{O}}_{\mathbf{w}}^J$ . Now  $c : \tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset} \rightarrow \mathbf{T}^r$  is compatible with the  $\mathbf{T}^{r-1}$ -actions on  $\tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset}, \mathbf{T}^r$  hence it induces a map  $'c : '\tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset} \rightarrow '\mathbf{T}^r$ . The homomorphism  $c' : \mathbf{T}^r \rightarrow \mathbf{T}$  given by

$$(t_1, t_2, \dots, t_r) \mapsto t_1 w_2(t_2) w_2 w_3(t_3) \dots w_2 w_3 \dots w_r(t_r)$$

is constant on each orbit of the  $\mathbf{T}^{r-1}$ -action on  $\mathbf{T}^r$  hence it induces a morphism  $'\mathbf{T}^r \rightarrow \mathbf{T}$  whose composition with  $'c$  is denoted by  $\bar{c} : '\tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset} \rightarrow \mathbf{T}$ . Let  $'M_{\lambda}^{\omega, \emptyset}$  be the local system  $\bar{c}^* L_{\lambda_1}$  on  $'\tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset}$  extended by 0 on  $'\tilde{\mathcal{B}}^{r+1} - '\tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset}$ . Let  $'M_{\lambda}^{\omega, J} \in \mathcal{D}_m(' \tilde{\mathcal{B}}^{r+1})$  be the intersection cohomology complex of  $'\tilde{\mathcal{O}}_{\mathbf{w}}^J$  with coefficients in  $'M_{\lambda}^{\omega, \emptyset}$  extended by 0 on  $'\tilde{\mathcal{B}}^{r+1} - '\tilde{\mathcal{O}}_{\mathbf{w}}^J$ . Let  $\bar{p}_{0r} : '\tilde{\mathcal{O}}_{\mathbf{w}}^J \rightarrow \tilde{\mathcal{B}}^2$  be the map induced by  $p_{0r} : \tilde{\mathcal{O}}_{\mathbf{w}}^J \rightarrow \tilde{\mathcal{B}}^2$ . We define  $'L_{\lambda}^{\omega, J} \in \mathcal{D}_m^{\bullet} \tilde{\mathcal{B}}^2$  as follows:

if  $\lambda_k = w_{k+1}(\lambda_{k+1})$  for  $k = 1, 2, \dots, r-1$ , we set  $'L_{\lambda}^{\omega, J} = \bar{p}_{0r}^* 'M_{\lambda}^{\omega, J} \langle |\mathbf{w}| \rangle$ ;  
otherwise, we set  $'L_{\lambda}^{\omega, J} = 0$ .

**3.10.** For  $L, L' \in \mathcal{C}_0^{\mathfrak{c}} \tilde{\mathcal{B}}^2$  we set

$$L \underline{\circ} L' = \underline{(L \circ L')}^{\{a-\nu\}} \in \mathcal{C}_0^{\mathfrak{c}} \tilde{\mathcal{B}}^2.$$

(For the notation  $\{i\}$  see 0.2.) By [L16, 2.24],  $L, L' \mapsto L \underline{\circ} L'$  defines a monoidal structure on  $\mathcal{C}_0^{\mathfrak{c}} \tilde{\mathcal{B}}^2$ . Hence if  ${}^1 L, {}^2 L, \dots, {}^r L$  are in  $\mathcal{C}_0^{\mathfrak{c}} \tilde{\mathcal{B}}^2$  then  ${}^1 L \underline{\circ} {}^2 L \underline{\circ} \dots \underline{\circ} {}^r L \in \mathcal{C}_0^{\mathfrak{c}} \tilde{\mathcal{B}}^2$  is well defined.

**3.11.** Let  $w \cdot \lambda \in I_n$  and let  $\omega \in \kappa^{-1}(w), s \in \mathbf{Z}$ . We show that we have canonically:

$$(a) \quad (\mathbf{e}^s)^* L_{\lambda}^{\omega} = L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\omega)}, \quad (\mathbf{e}^s)^* \mathbf{L}_{\lambda}^{\omega} = \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\omega)}.$$

It is enough to prove the first of these equalities. Let  $\xi = (x\mathbf{U}, y\mathbf{U}) \in \tilde{\mathcal{B}}^2$ . We have  $x^{-1}y \in \mathbf{U} \mathbf{e}^{-s}(\omega) t \mathbf{U}$  with  $t \in \mathbf{T}$  hence  $\mathbf{e}^s(x)^{-1} \mathbf{e}^s(y) \in \mathbf{U} \omega \mathbf{e}^s(t) \mathbf{U}$ . The stalk of  $(\mathbf{e}^s)^* L_{\lambda}^{\omega}$  at  $\xi$  is equal to the stalk of  $L_{\lambda}$  at  $\mathbf{e}^s(t)$  hence to the stalk of  $(\mathbf{e}^s)^* L_{\lambda}$  at  $t$ . The stalk of  $L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\omega)}$  at  $\xi$  is equal to the stalk of  $L_{\mathbf{e}^{-s}(\lambda)}$  at  $t$ . It remains to show that  $(\mathbf{e}^s)^* L_{\lambda} = L_{\mathbf{e}^{-s}(\lambda)}$ . This follows from the definitions.

#### 4. SHEAVES ON $Z_s$

**4.1.** In this section we fix  $s \in \mathbf{Z}$ .

Now  $\mathbf{T}$  acts on  $\tilde{\mathcal{B}}^2$  by  $t : (x\mathbf{U}, y\mathbf{U}) \mapsto (xt\mathbf{U}, ye^s(t)\mathbf{U})$ . Let  $\mathbf{T} \backslash_s \tilde{\mathcal{B}}^2$  be the set of orbits. Let

$$Z_s = \{(B, B', \gamma U_B); B \in \mathcal{B}, B' \in \mathcal{B}, \gamma U_B \in \tilde{G}_s / U_B; \gamma B \gamma^{-1} = B'\}.$$

We define  $\epsilon_s : \tilde{\mathcal{B}}^2 \rightarrow Z_s$  by  $\epsilon_s : (x\mathbf{U}, y\mathbf{U}) \mapsto (x\mathbf{B}x^{-1}, y\mathbf{B}y^{-1}, y\tau^s\mathbf{U}x^{-1})$ . Clearly,  $\epsilon_s$  induces a map  $\mathbf{T} \backslash_s \tilde{\mathcal{B}}^2 \rightarrow Z_s$ . We show:

(a)  $\epsilon_s$  induces an isomorphism  $\mathbf{T} \backslash_s \tilde{\mathcal{B}}^2 \rightarrow Z_s$ .

We show only that our map is bijective. Let  $(B, B', \gamma) \in \mathcal{B} \times \mathcal{B} \times \tilde{G}_s$  be such that  $\gamma B \gamma^{-1} = B'$ . We can find  $x \in G$  such that  $B = x\mathbf{B}x^{-1}$ . We set  $y = \gamma x \tau^{-s} \in G$ . Then  $\epsilon_s$  carries the  $\mathbf{T}$ -orbit of  $(x\mathbf{U}, y\mathbf{U})$  to  $(B, \gamma B \gamma^{-1}, \gamma x \mathbf{U} x^{-1}) = (B, B', \gamma U_B)$ ; thus our map is surjective. Now assume that  $x, x', y, y'$  in  $G$  are such that

$$(x\mathbf{B}x^{-1}, y\mathbf{B}y^{-1}, y\tau^s\mathbf{U}x^{-1}) = (x'\mathbf{B}x'^{-1}, y'\mathbf{B}y'^{-1}, y'\tau^s\mathbf{U}x'^{-1}).$$

To complete the proof of (a) it is enough to show that  $x' = xtu$ ,  $y' = ye^s(t)u'$  for some  $u, u'$  in  $\mathbf{U}$  and some  $t \in \mathbf{T}$ . Since  $x^{-1}x' \in \mathbf{B}$  we have  $x' = xtu$  for some  $u \in \mathbf{U}$  and some  $t \in \mathbf{T}$ . We have  $y'\tau^s\mathbf{U}u^{-1}t^{-1}x^{-1} = y\tau^s\mathbf{U}x^{-1}$  hence  $y' = ye^s(t)u'$  for some  $u' \in \mathbf{U}$ . This completes the proof of (a).

For  $w \in W$  let  $Z_s^w = \{(B, B', \gamma U_B) \in Z_s; (B, B') \in \mathcal{O}_w\}$ . The closure of  $Z_s^w$  in  $Z_s$  is  $\bar{Z}_s^w = \{(B, B', g U_B); (B, B') \in \bar{\mathcal{O}}_w, g \in G, g B g^{-1} = B'\}$ . We have  $\epsilon_s^{-1}(Z_s^w) = \tilde{\mathcal{O}}_w$ ,  $\epsilon_s^{-1}(\bar{Z}_s^w) = \tilde{\mathcal{O}}_w$ .

Let  $\omega \in \kappa_0^{-1}(w)$  and let  $\lambda \in \mathfrak{s}_\infty$  be such that  $w \cdot \lambda \in I^s$ . We have a diagram  $\mathbf{T} \xleftarrow{j^\omega} \tilde{\mathcal{B}}_w^2 \xrightarrow{\epsilon_s^w} Z_s^w$  where  $\epsilon_s^w$  is the restriction of  $\epsilon_s$  and  $j^\omega$  is as in 3.1. The  $\mathbf{T}$ -action on  $\tilde{\mathcal{B}}^2$  described above is compatible under  $j^\omega$  with the  $\mathbf{T}$ -action on  $\mathbf{T}$  given by  $t : t' \mapsto w^{-1}(t^{-1})t'e^s(t)$ . From [L9, 28.2] we see that  $L_\lambda$  is equivariant for the  $\mathbf{T}$ -action on  $\mathbf{T}$  given by  $t : t' \mapsto w^{-1}(\mathbf{e}^{-s}(t_1))t't_1^{-1}$ . (We use that  $w \cdot \lambda \in I^s$ .) Using the change of variable  $t_1 = \mathbf{e}^s(t)^{-1}$ , we deduce that  $L_\lambda$  is also equivariant for the  $\mathbf{T}$ -action on  $\mathbf{T}$  given by  $t : t' \mapsto w^{-1}(t^{-1})t'e^s(t)$ . It follows that  $(j^\omega)^* L_\lambda$  is  $\mathbf{T}$ -equivariant, so that there is a well defined local system  $\mathcal{L}_{\lambda,s}^\omega$  of rank 1 on  $Z_s^w$  such that  $(\epsilon_s^w)^* \mathcal{L}_{\lambda,s}^\omega = (j^\omega)^* L_\lambda = L_\lambda^\omega$ . Let  $\mathcal{L}_{\lambda,s}^{\omega\sharp}$  be its extension to an intersection cohomology complex of  $\bar{Z}_s^w$ , viewed as a complex on  $Z_s$ , equal to 0 on  $Z_s - \bar{Z}_s^w$ . We shall view  $\mathcal{L}_{\lambda,s}^\omega$  as a constructible sheaf on  $Z_s$  which is 0 on  $Z_s - Z_s^w$ . Let

$$\mathbb{L}_{\lambda,s}^\omega = \mathcal{L}_{\lambda,s}^{\omega\sharp} \langle |w| + \nu + \rho \rangle,$$

a simple perverse sheaf on  $Z_s$ .

*In the remainder of this subsection we assume that  $s \neq 0$  and that we are in case A.*

Let  $w \in W$  and let  $X_s^w = \{B \in \mathcal{B}; (B, \mathbf{e}^s(B)) \in \mathcal{O}_w\}$ . When  $s > 0$ ,  $X_s^w$  coincides with the variety  $X_w$  defined in [DL] in terms of the Frobenius map  $\mathbf{e}^s : G \rightarrow G$ ; when  $s < 0$ ,  $X_s^w$  can be identified with the variety  $X_{\mathbf{e}^{-s}(w^{-1})}$  defined in [DL] in terms of the Frobenius map  $\mathbf{e}^{-s} : G \rightarrow G$ . Note that the finite group  $G^{\mathbf{e}^s} = \{g \in G; \mathbf{e}^s(g) = g\}$  acts by conjugation on  $X_s^w$ .

Let  $\tilde{X}_s^w = \{x\mathbf{U} \in G/\mathbf{U}; x^{-1}\mathbf{e}^s(x) \in G_w\}$ . We define  $\phi : \tilde{X}_s^w \rightarrow X_s^w$  by  $x\mathbf{U} \mapsto x\mathbf{B}x^{-1}$ . This is a principal  $\mathbf{T}$ -bundle with  $\mathbf{T}$  acting on  $\tilde{X}_s^w$  by  $t : x\mathbf{U} \mapsto xt\mathbf{U}$ . We define  $j'_w : \tilde{X}_s^w \rightarrow \mathbf{T}$  by  $j'_w(x\mathbf{U}) = (x^{-1}\mathbf{e}^s(x))_w$ . Now let  $\lambda \in \mathfrak{s}_\infty$  be such that  $w \cdot \lambda \in I^s$ . Then there is a well defined local system  $\mathcal{F}_{\lambda,s}^w$  on  $X_s^w$  such

that  $\phi^* \mathcal{F}_{\lambda,s}^{\dot{w}} = (j'_w)^* L_\lambda$ . (This is in fact the restriction of  $\mathcal{L}_{\lambda,s}^{\dot{w}}$  to  $X_s^w$  under the imbedding  $X_s^w \rightarrow Z_s^w$ ,  $x\mathbf{B}x^{-1} \mapsto (x\mathbf{B}x^{-1}, \mathbf{e}^s(x)\mathbf{B}\mathbf{e}^s(x^{-1}), \tau^s x \mathbf{U} x^{-1})$ .) The local system  $\mathcal{F}_{\lambda,s}^{\dot{w}}$  on  $X_s^w$  is of the type considered in [DL]. Note also that  $\mathcal{F}_{\lambda,s}^{\dot{w}}$  has a natural  $G^{\mathbf{e}^s}$ -equivariant structure. (It is the restriction of the  $G$ -equivariant structure of  $\mathcal{L}_{\lambda,s}^{\dot{w}}$ .) It follows that for  $j \in \mathbf{Z}$ ,  $H_c^j(X_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})$  is naturally a  $G^{\mathbf{e}^s}$ -module. (This representation of  $G^{\mathbf{e}^s}$  is one of the main themes of [DL].) Let  $\bar{X}_s^w = \{B \in \mathcal{B}; (B, \mathbf{e}^s(B)) \in \bar{\mathcal{O}}_w\}$ . Then  $X_s^w$  is open dense smooth in  $\bar{X}_s^w$  and  $G^{\mathbf{e}^s}$  acts by conjugation on  $\bar{X}_s^w$ . Hence for  $j \in \mathbf{Z}$ , the intersection cohomology space  $IH^j(\bar{X}_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})$  is naturally a  $G^{\mathbf{e}^s}$ -module.

If  $\mathbf{r}, \mathbf{r}'$  are  $G^{\mathbf{e}^s}$ -modules and  $\mathbf{r}$  is irreducible we denote by  $(\mathbf{r} : \mathbf{r}')$  the multiplicity of  $\mathbf{r}$  in  $\mathbf{r}'$ . Let  $\text{Irr}(G^{\mathbf{e}^s})$  be the set of isomorphism classes of irreducible representations of  $G^{\mathbf{e}^s}$ . From [DL, 7.7] it is known that for any  $\mathbf{r} \in \text{Irr}(G^{\mathbf{e}^s})$

(i) there exists  $w \cdot \lambda \in I^s$  such that  $(\mathbf{r} : \oplus_j H_c^j(X_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})) \neq 0$ .

From [L1, 2.8] we see using (i) that for any  $\mathbf{r} \in \text{Irr}(G^{\mathbf{e}^s})$

(ii) there exists  $w \cdot \lambda \in I^s$  such that  $(\mathbf{r} : \oplus_j IH^j(X_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})) \neq 0$ .

By [DL, 6.3], any  $\mathbf{r} \in \text{Irr}(G^{\mathbf{e}^s})$  determines a  $W$ -orbit  $\mathfrak{o}$  on  $\mathfrak{s}_\infty$ : the set of all  $\lambda \in \mathfrak{s}_\infty$  such that  $(\mathbf{r} : \oplus_j H_c^j(X_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})) \neq 0$  for some  $w \in W$  with  $w \cdot \lambda \in I^s$  or equivalently (see [L1, 2.8]) such that  $(\mathbf{r} : \oplus_j IH^j(\bar{X}_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})) \neq 0$  for some  $w \in W$  with  $w \cdot \lambda \in I^s$ ; we have necessarily  $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$ . For any  $\mathfrak{o} \in W \backslash \mathfrak{s}_\infty$  such that  $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$ , let  $\text{Irr}_{\mathfrak{o}}(G^{\mathbf{e}^s})$  be the set of all  $\mathbf{r} \in \text{Irr}(G^{\mathbf{e}^s})$  such that the  $W$ -orbit on  $\mathfrak{s}_\infty$  determined by  $\mathbf{r}$  is  $\mathfrak{o}$ . With notation in 1.14 we have the following result:

(b) *There exists a pairing  $\text{Irr}_{\mathfrak{o}}(G^{\mathbf{e}^s}) \times \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1) \rightarrow \bar{\mathbf{Q}}_l$ ,  $(\mathbf{r}, E) \mapsto b_{\mathbf{r},E}$  such that for any  $\mathbf{r} \in \text{Irr}_{\mathfrak{o}}(G^{\mathbf{e}^s})$ , any  $z \cdot \lambda \in I^s \cap I_{\mathfrak{o}}$  and any  $j \in \mathbf{Z}$  we have*

$$(\mathbf{r} : IH^j(\bar{X}_s^z, \mathcal{F}_{\lambda,s}^{\dot{z}})) = (-1)^j(j - |z|) : \sum_{E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)} b_{\mathbf{r},E} \text{tr}(\mathbf{e}_s c_{z \cdot \lambda}, E^v).$$

In the case where  $G$  has connected centre, (b) is just a reformulation on [L1, 3.8(ii)]. A proof similar to that in *loc.cit.* applies without the hypothesis on the centre.

**4.2.** *In the remainder of this section let  $\mathbf{c}, a, \mathfrak{o}, n, \Psi$  be as in 3.1(a).*

The  $G \times \mathbf{T}^2$ -action on  $\tilde{\mathcal{B}}^2$  defined in 3.1 commutes with the  $\mathbf{T}$ -action on  $\tilde{\mathcal{B}}^2$  in 4.1; hence it induces a  $G \times \mathbf{T}^2$ -action on  $\mathbf{T} \backslash_s \tilde{\mathcal{B}}^2$ . We define a  $G \times \mathbf{T}^2$ -action on  $Z_s$  by

$$(g, t_1, t_2) : (B, B', \gamma U_B) \mapsto (gBg^{-1}, gB'g^{-1}, g\gamma x_0 \mathbf{e}^s(t_2^{-n}) t_1^n x_0^{-1} g^{-1} U_{gBg^{-1}})$$

where  $x_0$  is any element of  $G$  such that  $x_0 \mathbf{B} x_0^{-1} = B$ . (The choice of  $x_0$  does not matter; to see this, it is enough to show that for  $b \in B$  we have

$$\gamma x_0 \mathbf{e}^s(t_2^{-n}) t_1^n x_0^{-1} U_B = \gamma x_0 b \mathbf{e}^s(t_2^{-n}) t_1^n b^{-1} x_0^{-1} U_B$$

which is immediate.) In this  $G \times \mathbf{T}^2$  action, the subgroup  $\{(1, t_1, t_2) \in G \times \mathbf{T}^2; t_1 = \mathbf{e}^s(t_2)\}$  acts trivially. Note that the bijection  $\mathbf{T} \backslash_s \tilde{\mathcal{B}}^2 \rightarrow Z_s$  in 4.1(a) is compatible with the  $G \times \mathbf{T}^2$ -actions.



Let  $w \in W, \omega \in \kappa_0^{-1}(w)$ . Since the  $G \times \mathbf{T}^2$ -action on  $\tilde{\mathcal{O}}_w$  is transitive, it follows that the  $G \times \mathbf{T}^2$ -action on  $Z_s^w$  is transitive. We show :

(a) *Let  $\mathcal{L}$  be an irreducible  $G \times \mathbf{T}^2$ -equivariant local system on  $Z_s^w$ . Then  $\mathcal{L}$  is isomorphic to  $\mathcal{L}_{\lambda,s}^\omega$  for a unique  $\lambda \in \mathfrak{s}_n$  such that  $w \cdot \lambda \in I^s$ .*

The local system  $(\epsilon_s^w)^* \mathcal{L}$  on  $\tilde{\mathcal{O}}_w$  is irreducible and  $G \times \mathbf{T}^2$ -equivariant hence, by 3.1(c), is isomorphic to  $L_\lambda^\omega$  for a well defined  $\lambda \in \mathfrak{s}_n$ . Now the restriction of  $(\epsilon_s^w)^* \mathcal{L}$  to any fibre of  $\epsilon_s^w$  is  $\mathbf{Q}_l$ . On the other hand, the restriction of  $L_\lambda^\omega$  to the fibre of  $\epsilon_s^w$  passing through  $(\mathbf{U}, \omega \mathbf{U})$  is (under an obvious identification with  $\mathbf{T}$ ) the inverse image of  $L_\lambda$  under the map  $\mathbf{T} \rightarrow \mathbf{T}, t \mapsto w^{-1}(t^{-1})\mathbf{e}^s(t)$ , hence it is  $L_{w(\lambda^{-1})\mathbf{e}^{-s}(\lambda)}$  which is  $\mathbf{Q}_l$  if and only if  $w(\lambda) = \mathbf{e}^{-s}\lambda$ . We see that we must have  $w(\lambda) = \mathbf{e}^{-s}(\lambda)$ . We have  $(\epsilon_s^w)^* \mathcal{L} \cong (\epsilon_s^w)^* \mathcal{L}_{\lambda,s}^\omega$  (both are  $L_\lambda^\omega$ ) hence  $\mathcal{L} \cong \mathcal{L}_{\lambda,s}^\omega$ . This proves (a).

We define  $\mathfrak{h} : Z_s \rightarrow Z_{-s}$  by  $(B, B', gU_B) \mapsto (B', B, g^{-1}U_{B'})$ . Note that  $\mathfrak{h}\epsilon_s = \epsilon_{-s}\tilde{\mathfrak{h}} : \tilde{\mathcal{B}}^2 \rightarrow Z_{-s}$  with  $\tilde{\mathfrak{h}}$  as in 3.1. For  $L \in \mathcal{D}_m(Z_{-s})$  we set  $L^\dagger = \mathfrak{h}^* L$ .

**4.3.** Let

$$I_n^s = I_n \cap I^s.$$

Note that if  $w \cdot \lambda \in I_n^s$  and  $\omega \in \kappa_0^{-1}(w)$ , then  $\mathcal{L}_{\lambda,s}^\omega|_{Z_s^w}, \mathbb{L}_{\lambda,s}^\omega$  can be regarded naturally as objects in the mixed derived category of pure weight zero. Moreover,  $\mathcal{L}_{\lambda,s}^\omega|_{Z_s^w}$  (resp.  $\mathbb{L}_{\lambda,s}^\omega$ ) is (noncanonically) isomorphic to  $\mathcal{L}_{\lambda,s}^{\dot{w}}|_{Z_s^w}$  (resp.  $\mathbb{L}_{\lambda,s}^{\dot{w}}$ ) in the mixed derived category.

We define  $\tilde{\epsilon}_s : \mathcal{D}(Z_s) \rightarrow \mathcal{D}(\tilde{\mathcal{B}}^2), \tilde{\epsilon}_s : \mathcal{D}_m(Z_s) \rightarrow \mathcal{D}_m(\tilde{\mathcal{B}}^2)$  by

$$\tilde{\epsilon}_s(L) = \epsilon_s^*(L)\langle \rho \rangle.$$

From the definition we have

$$\epsilon_s^* \mathcal{L}_{\lambda,s}^{\omega\sharp} = L_\lambda^{\omega\sharp}, \quad \tilde{\epsilon}_s \mathbb{L}_{\lambda,s}^\omega = \mathbf{L}_\lambda^\omega.$$

Let  $\mathcal{D}^\spadesuit Z_s$  be the subcategory of  $\mathcal{D}(Z_s)$  consisting of objects which are restrictions of objects in the  $G \times \mathbf{T}^2$ -equivariant derived category. Let  $\mathcal{M}^\spadesuit Z_s$  be the subcategory of  $\mathcal{D}^\spadesuit Z_s$  consisting of objects which are perverse sheaves. Let  $\mathcal{M}^\preceq Z_s$  (resp.  $\mathcal{M}^\prec Z_s$ ) be the subcategory of  $\mathcal{D}^\spadesuit Z_s$  whose objects are perverse sheaves  $L$  such that any composition factor of  $L$  is of the form  $\mathbb{L}_{\lambda,s}^{\dot{w}}$  for some  $w \cdot \lambda \in I_n^s$  such that  $w \cdot \lambda \preceq \mathbf{c}$  (resp.  $w \cdot \lambda \prec \mathbf{c}$ ). Let  $\mathcal{D}^\preceq Z_s$  (resp.  $\mathcal{D}^\prec Z_s$ ) be the subcategory of  $\mathcal{D}^\spadesuit Z_s$  whose objects are complexes  $L$  such that  $L^j$  is in  $\mathcal{M}^\preceq Z_s$  (resp.  $\mathcal{M}^\prec Z_s$ ) for any  $j$ . We write  $\mathcal{D}_m()$  or  $\mathcal{M}_m()$  for the mixed version of any of the categories above.

Let  $\mathcal{C}^\spadesuit Z_s$  be the subcategory of  $\mathcal{M}^\spadesuit Z_s$  consisting of semisimple objects. Let  $\mathcal{C}_0^\spadesuit Z_s$  be the subcategory of  $\mathcal{M}_m^\spadesuit Z_s$  consisting of objects of pure of weight zero. Let  $\mathcal{C}^c Z_s$  be the subcategory of  $\mathcal{M}^\spadesuit Z_s$  consisting of objects which are direct sums of objects of the form  $\mathbb{L}_{\lambda,s}^{\dot{w}}$  with  $w \cdot \lambda \in \mathbf{c}^s$ . Let  $\mathcal{C}_0^c Z_s$  be the subcategory of  $\mathcal{C}_0^\spadesuit Z_s$  consisting of those  $L \in \mathcal{C}_0^\spadesuit Z_s$  such that, as an object of  $\mathcal{C}^\spadesuit Z_s$ ,  $L$  belongs to  $\mathcal{C}^c Z_s$ . For  $L \in \mathcal{C}_0^\spadesuit Z_s$  let  $\underline{L}$  be the largest subobject of  $L$  such that as an object of  $\mathcal{C}^\spadesuit Z_s$ , we have  $\underline{L} \in \mathcal{C}^c Z_s$ .

From 4.2(a) we see that, if  $M \in \mathcal{M}^\bullet Z_s$ , then any composition factor of  $M$  is of the form  $\mathbb{L}_{\lambda,s}^{\dot{w}}$  for some  $w \cdot \lambda \in I_n^s$ . From the definitions we see that  $M \mapsto \tilde{\epsilon}_s M$  is a functor  $\mathcal{D}^\bullet Z_s \rightarrow \mathcal{D}^\bullet \tilde{\mathcal{B}}^2$  and also  $\mathcal{D}_m^\bullet Z_s \rightarrow \mathcal{D}_m^\bullet \tilde{\mathcal{B}}^2$ ; moreover, it is a functor  $\mathcal{M}^\bullet Z_s \rightarrow \mathcal{M}^\bullet \tilde{\mathcal{B}}^2$  and also  $\mathcal{M}_m^\bullet Z_s \rightarrow \mathcal{M}_m^\bullet \tilde{\mathcal{B}}^2$ . From the definitions we see that for  $M \in \mathcal{M}^\bullet Z_s$

(a) *we have  $M \in \mathcal{M}^\preceq Z_s$  if and only if  $\tilde{\epsilon}_s M \in \mathcal{M}^\preceq \tilde{\mathcal{B}}^2$ ; we have  $M \in \mathcal{M}^\prec Z_s$  if and only if  $\tilde{\epsilon}_s M \in \mathcal{M}^\prec \tilde{\mathcal{B}}^2$ .*

Note that if  $X \in \mathcal{D}(Z_s)$  and  $j \in \mathbf{Z}$ , then

$$(b) \quad (\epsilon_s^* X)^{j+\rho} = \epsilon_s^*(X^j)[\rho].$$

Moreover, if  $Y \in \mathcal{M}_m(Z_s)$  and  $j' \in \mathbf{Z}$  then

$$(c) \quad gr_{j'}(\tilde{\epsilon}_s Y) = \tilde{\epsilon}_s(gr_{j'} Y).$$

For  $w \cdot \lambda \in I_n$  we show:

(d) *We have  $w \cdot \lambda \in I_n^s$  if and only if  $w^{-1} \cdot w(\lambda^{-1}) \in I_n^{-s}$ .*

We must show that we have  $w(\lambda) = \mathbf{e}^{-s}(\lambda)$  if and only if  $\lambda^{-1} = \mathbf{e}^s(w(\lambda^{-1}))$ . In other words, we must show that  $\lambda(w^{-1}(t)) = \lambda(\tau^s t \tau^{-s})$  for all  $t \in \mathbf{T}_n$  if and only if  $\lambda(t') = \lambda(w^{-1}(\tau^{-s} t' \tau^s))$  for all  $t' \in \mathbf{T}_n$ . Setting  $t' = \tau^s t \tau^{-s}$ , we have  $w^{-1}(t) = w^{-1}(\tau^{-s} t' \tau^s)$  and it remains to use that  $t \mapsto \tau^s t \tau^{-s}$  is a bijection  $\mathbf{T}_n \rightarrow \mathbf{T}_n$ .

For  $w \cdot \lambda \in I_n^s$  we show:

(e) *Let  $\omega \in \kappa_0^{-1}(w)$ . We have canonically  $(\mathbb{L}_{\lambda,s}^\omega)^\dagger = \mathbb{L}_{w(\lambda^{-1}),-s}^{\omega^{-1}}$ .*

(The equality in (e) makes sense in view of (d).) By [L16, 2.2(a)] and with notation of 3.1 we have canonically  $\tilde{\mathfrak{h}}^* \mathbf{L}_\lambda^\omega = \mathbf{L}_{w(\lambda^{-1})}^{\omega^{-1}}$ . Hence  $\epsilon_{-s}^* \mathbf{L}_{w(\lambda^{-1})}^{\omega^{-1}} = \epsilon_{-s}^* \tilde{\mathfrak{h}}^* \mathbf{L}_\lambda^\omega = \mathfrak{h}^* \epsilon_s^* \mathbf{L}_\lambda^\omega$  so that  $\tilde{\epsilon}_{-s} \mathbf{L}_{w(\lambda^{-1})}^{\omega^{-1}} = \mathfrak{h}^* \tilde{\epsilon}_s \mathbf{L}_\lambda^\omega$  and (e) follows.

**4.4.** Let  $r, f$  be integers such that  $0 \leq f \leq r-3$ . Let

$\mathcal{Y}$

$$= \{((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^{r+1} \times \tilde{G}_s; \gamma \in x_{f+3} \mathbf{U} \tau^s x_f^{-1}, \gamma \in x_{f+2} \mathbf{B} \tau^s x_{f+1}^{-1}\}.$$

Define  $\vartheta : \mathcal{Y} \rightarrow \tilde{\mathcal{B}}^{r+1}$  by  $((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}), \gamma) \mapsto (x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U})$ . For  $y', y'' \in W$  let

$$\tilde{\mathcal{B}}_{[y', y'']}^{r+1} = \{(x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}) \in \tilde{\mathcal{B}}^{r+1}; x_f^{-1} x_{f+1} \in G_{y'}, x_{f+2}^{-1} x_{f+3} \in G_{y''^{-1}}\}.$$

We show:

(a) *Let  $\xi \in \tilde{\mathcal{B}}_{[y', y'']}^{r+1}$ . If  $\mathbf{e}^s(y') \neq y''$  then  $\vartheta^{-1}(\xi) = \emptyset$ . If  $\mathbf{e}^s(y') = y''$  then  $\vartheta^{-1}(\xi) \cong \mathbf{k}^{\nu - |y'|}$ .*

We set  $\xi = (x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U})$ . If  $\vartheta^{-1}(\xi) \neq \emptyset$  then  $x_f^{-1} x_{f+1} \in G_{y'}$ ,  $x_{f+2}^{-1} x_{f+3} \in$

$G_{y''-1}$  and  $(x_{f+3}\mathbf{U}\tau^s x_f^{-1}) \cap (x_{f+2}\mathbf{B}\tau^s x_{f+1}^{-1}) \neq \emptyset$ . Hence for some  $u \in \mathbf{U}$ ,  $b \in \mathbf{B}$  we have

$$u\tau^s x_f^{-1} x_{f+1} = x_{f+3}^{-1} x_{f+2} b \tau^s \in \tau^s G_{y'} \cap G_{y''} \tau^s$$

so that  $\mathbf{e}^s(y') = y''$ . If we assume that  $\mathbf{e}^s(y') = y''$ , then  $\vartheta^{-1}(\xi)$  can be identified with

$$\{\gamma \in \tilde{G}_s; \gamma \in x_{f+3}\mathbf{U}\tau^s x_f^{-1}, \gamma \in x_{f+2}\mathbf{B}\tau^s x_{f+1}^{-1}\}$$

hence with

$$\{(u, b) \in \mathbf{U} \times \mathbf{B}; u\tau^s x_f^{-1} x_{f+1} = x_{f+3}^{-1} x_{f+2} b \tau^s\}.$$

We substitute  $x_{f+3}^{-1} x_{f+2} = u_0 \mathbf{e}^s(\dot{y}') t_0 u'_0$ ,  $x_f^{-1} x_{f+1} = u_1 \dot{y}' t_1 u'_1$ , where  $t_0 \in \mathbf{T}$ ,  $u_0, u'_0, u_1, u'_1 \in \mathbf{U}$ . Then  $\vartheta^{-1}(\xi)$  is identified with  $\{(u, b) \in \mathbf{U} \times \mathbf{B}; u\tau^s u_1 \dot{y}' t_1 u'_1 = u_0 \mathbf{e}^s(\dot{y}') t_0 u'_0 b \tau^s\}$ . The map  $(u, b) \mapsto u_0^{-1} u \mathbf{e}^s(u_1)$  identifies this variety with  $\mathbf{U} \cap \mathbf{e}^s(\dot{y}') \mathbf{B} \mathbf{e}^s(\dot{y}')^{-1} \cong \mathbf{k}^{\nu-|\dot{y}'|}$ . This proves (a).

Now  $\mathbf{T}^2$  acts freely on  $\mathcal{Y}$  by

$$\begin{aligned} (t_1, t_2) : ((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}), \gamma) &\mapsto \\ ((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_f \mathbf{U}, x_{f+1} t_1 \mathbf{U}, x_{f+2} t_2 \mathbf{U}, x_{f+3} \mathbf{U}, \dots, x_r \mathbf{U}), \gamma). \end{aligned}$$

Let

$$\begin{aligned} {}^1\mathcal{Y} = \mathbf{T} \setminus \{ &((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^{r+1} \times \tilde{G}_s; \\ &\gamma \in x_{f+3} \mathbf{U} \tau^s x_f^{-1}, \gamma \in x_{f+2} \mathbf{U} \tau^s x_{f+1}^{-1} \} \end{aligned}$$

where  $\mathbf{T}$  acts freely by

$$\begin{aligned} t : ((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}), \gamma) &\mapsto \\ ((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_f \mathbf{U}, x_{f+1} \mathbf{e}^{-s}(t) \mathbf{U}, x_{f+2} t \mathbf{U}, x_{f+3} \mathbf{U}, \dots, x_r \mathbf{U}), \gamma). \end{aligned}$$

Note that the obvious map  $\beta : {}^1\mathcal{Y} \rightarrow \mathbf{T}^2 \setminus \mathcal{Y}$  is an isomorphism. We define  ${}^1\eta : {}^1\mathcal{Y} \rightarrow Z_s$  by

$$((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}), \gamma) \mapsto \epsilon_s(x_{f+1} \mathbf{U}, x_{f+2} \mathbf{U}).$$

We define  $\tau : \mathcal{Y} \rightarrow {}^1\mathcal{Y}$  as the composition of the obvious map  $\mathcal{Y} \rightarrow \mathbf{T}^2 \setminus \mathcal{Y}$  with  $\beta^{-1}$ . Let  $\eta = {}^1\eta \tau : \mathcal{Y} \rightarrow Z_s$ . We have

$$\eta((x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}), \gamma) = \epsilon_s(x_{f+1} t^{-1} \mathbf{U}, x_{f+2} t'^{-1} \mathbf{U})$$

where  $t, t'$  in  $\mathbf{T}$  are such that  $\gamma \in x_{f+2} t'^{-1} \mathbf{U} \tau^s t x_{f+1}^{-1}$ .

**4.5.** Let  $z \cdot \lambda \in I_n^s$ . Let  $P = \eta^* \mathcal{L}_{\lambda, s}^{\dot{z}\sharp}$ . Let  $p_{ij} : \tilde{\mathcal{B}}^{r+1} \rightarrow \tilde{\mathcal{B}}^2$  be the projection to the  $ij$  coordinates. We have the following result:

$$(a) \quad \vartheta_! P \simeq \{p_{f, f+1}^* L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{f+1, f+2}^* L_{\lambda}^{\dot{z}\sharp} \otimes p_{f+2, f+3}^* L_{y(\lambda)}^{\dot{y}^{-1}} \langle 2|y| - 2\nu \rangle; y \in W\}.$$

Define  $e : \tilde{\mathcal{B}}^{r+1} \rightarrow \tilde{\mathcal{B}}^4$  by

$$(x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}) \mapsto (x_f \mathbf{U}, x_{f+1} \mathbf{U}, x_{f+2} \mathbf{U}, x_{f+3} \mathbf{U}).$$

Then (a) is obtained by applying  $e^*$  to the statement similar to (a) in which  $\{0, 1, \dots, r\}$  is replaced by  $\{f, f+1, f+2, f+3\}$ . Thus it is enough to prove (a) in the special case where  $r = 3, f = 0$ . In the remainder of the proof we assume that  $r = 3, f = 0$ .

For any  $y', y''$  in  $W$  let  $\vartheta_{y', y''} : \vartheta^{-1}(\tilde{\mathcal{B}}_{[y', y'']}^4) \rightarrow \tilde{\mathcal{B}}^4$  be the restriction of  $\vartheta$ . Let  $P^{y', y''}$  be the restriction of  $P$  to  $\vartheta^{-1}(\tilde{\mathcal{B}}^4)_{[y', y'']}$ . Clearly, we have

$$\vartheta_! P \simeq \{(\vartheta_{y', y''})_! P^{y', y''}; (y', y'') \in W^2\}.$$

Since  $\vartheta^{-1}(\tilde{\mathcal{B}}_{[y', y'']}^{r+1}) = \emptyset$  when  $\mathbf{e}^s(y') \neq y''$ , see 4.4(a), we deduce that

$$\vartheta_! P \simeq \{(\vartheta_{\mathbf{e}^{-s}(y), y^{-1}})_! P^{\mathbf{e}^{-s}(y), y^{-1}}; y \in W\}.$$

Hence to prove (a) it is enough to show for any  $y \in W$  that

$$\vartheta_y! P_y = p_{01}^* L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^* L_{\lambda}^{\dot{z}\sharp} \otimes p_{23}^* L_{y(\lambda)}^{\dot{y}^{-1}} \langle 2|y| - 2\nu \rangle,$$

where we write  $\vartheta_y, P_y$  instead of  $\vartheta_{\mathbf{e}^{-s}(y), y^{-1}}, P^{\mathbf{e}^{-s}(y), y^{-1}}$ . Using  $z(\lambda) = \mathbf{e}^{-s}(\lambda)$  we can replace  $p_{01}^* L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})}$  by  $p_{01}^* L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})}$ . Thus it is enough to show for any  $y \in W$  that

$$(b) \quad \vartheta_y! P_y = p_{01}^* L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^* L_{\lambda}^{\dot{z}\sharp} \otimes p_{23}^* L_{y(\lambda)}^{\dot{y}^{-1}} \langle 2|y| - 2\nu \rangle.$$

We have a cartesian diagram

$$\begin{array}{ccc} \tilde{V}_y & \xrightarrow{\tilde{b}} & \tilde{\mathcal{V}}_y \\ \downarrow & & \downarrow \\ V_y & \xrightarrow{b} & \mathcal{V}_y \end{array}$$

where

$$V_y = \{(x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}) \in \tilde{\mathcal{B}}^4; x_0^{-1} x_1 \in G_{\mathbf{e}^{-s}(y)}, x_1^{-1} x_2 \in G_z, x_2^{-1} x_3 \in G_{y^{-1}}\},$$

$$\mathcal{V}_y = \mathbf{T} \backslash \{ (x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}) \in \tilde{\mathcal{B}}^4; x_0^{-1} x_1 \in G_{\mathbf{e}^{-s}(y)}, x_1^{-1} x_2 \in G_z, \\ x_2^{-1} x_3 \in G_{y^{-1}}, \mathbf{e}^s((x_0^{-1} x_1)_{\mathbf{e}^{-s}(\dot{y})}) = (x_3^{-1} x_2)_{\dot{y}} \}$$

with  $\mathbf{T}$  acting (freely) by

$$t : (x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}) \mapsto (x_0 \mathbf{U}, x_1 \mathbf{e}^{-s}(t) \mathbf{U}, x_2 t \mathbf{U}, x_3 \mathbf{U}),$$

$\tilde{V}_y = \vartheta^{-1}(V_y)$  and

$$\tilde{\mathcal{V}}_y = \mathbf{T} \backslash \{ ((x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^4 \times \tilde{G}_s; x_0^{-1} x_1 \in G_{\mathbf{e}^{-s}(y)}, x_1^{-1} x_2 \in G_z, \\ x_2^{-1} x_3 \in G_{y^{-1}}, \gamma \in x_3 \mathbf{U} \tau^s x_0^{-1}, \gamma \in x_2 \mathbf{U} \tau^s x_1^{-1} \}$$

with  $\mathbf{T}$  acting (freely) by

$$t : ((x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}), \gamma) \mapsto ((x_0 \mathbf{U}, x_1 \mathbf{e}^{-s}(t) \mathbf{U}, x_2 t \mathbf{U}, x_3 \mathbf{U}), \gamma);$$

we have

$$b(x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}) = \mathbf{T} - \text{orbit of } (x_0 \mathbf{U}, x_1 t \mathbf{U}, x_2 t' \mathbf{U}, x_3 \mathbf{U})$$

where  $t, t'$  in  $\mathbf{T}$  are such that  $\mathbf{e}^s((x_0^{-1} x_1 t)_{\mathbf{e}^{-s}(\dot{y})}) = (x_3^{-1} x_2 t')_{\dot{y}}$ ,

$$\tilde{b}((x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}), \gamma) = \mathbf{T} - \text{orbit of } ((x_0 \mathbf{U}, x_1 t \mathbf{U}, x_2 t' \mathbf{U}, x_3 \mathbf{U}), \gamma)$$

where  $t, t'$  in  $\mathbf{T}$  are such that  $\gamma \in x_2 t' \mathbf{U} \tau^s t^{-1} x_1^{-1}$ ; the vertical maps are the obvious ones. We also have a cartesian diagram

$$\begin{array}{ccc} \tilde{V}'_y & \xrightarrow{\tilde{b}'} & \tilde{\mathcal{V}}'_y \\ \downarrow & & \downarrow \\ V'_y & \xrightarrow{b'} & \mathcal{V}'_y \end{array}$$

where  $\tilde{V}'_y, \tilde{\mathcal{V}}'_y, V'_y, \mathcal{V}'_y$  are defined in the same way as  $\tilde{V}_y, \tilde{\mathcal{V}}_y, V_y, \mathcal{V}_y$  but the condition  $x_1^{-1} x_2 \in G_z$  is replaced by the condition  $x_1^{-1} x_2 \in \bar{G}_z$ ; the maps  $\tilde{b}', b'$  are given by the same formulas as  $\tilde{b}, b$ ; the vertical maps are the obvious ones.

Let  $j : V'_y \rightarrow \tilde{\mathcal{B}}^4$  be the inclusion. It is enough to show that

$$j^* \vartheta_{y!} P_y = j^* (p_{01}^* L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^* L_{\lambda}^{\dot{z}\#} \otimes p_{23}^* L_{y(\lambda)}^{\dot{y}^{-1}}) \langle 2|y| - 2\nu \rangle.$$

By definition,  $P|_{\tilde{V}'_y}$  is the inverse image of  $\mathcal{L}_{\lambda, s}^{\dot{z}\#}$  under the composition of  $\tilde{b}'$  with

$\tilde{\mathcal{V}}'_y \rightarrow \mathcal{V}'_y \xrightarrow{! \eta_y} Z_s$  where the first map is the obvious one and

$$! \eta_y(x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}) = \epsilon_s(x_1 \mathbf{U}, x_2 \mathbf{U}).$$

Hence  $P|_{\tilde{V}'_y}$  is the inverse image of  $\mathcal{L}_{\lambda,s}^{\ddagger\sharp}$  under the composition of  $\eta_y := {}^!\eta_y b'$  with the obvious map  $\vartheta'_y : \tilde{V}'_y \rightarrow V'_y$ . Since  $\vartheta_y$  is an affine space bundle with fibres of dimension  $\nu - |y|$ , it follows that  $j^* \vartheta_{y!} P_y = \eta_y^* \mathcal{L}_{\lambda,s}^{\ddagger\sharp} \langle 2|y| - 2\nu \rangle$ . Thus it is enough to show that

$$\eta_y^* \mathcal{L}_{\lambda,s}^{\ddagger\sharp} = j^* (p_{01}^* L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^* L_{\lambda}^{\ddagger\sharp} \otimes p_{23}^* L_{y(\lambda)}^{\dot{y}^{-1}}).$$

Since  $\eta_y$  is smooth as a map to  $\bar{Z}_s^z$ , we see that  $\eta_y^* \mathcal{L}_{\lambda,s}^{\ddagger\sharp}$  is the intersection cohomology complex of  $V'_y$  with coefficients in the local system  $(\eta_y^0)^* \mathcal{L}_{\lambda,s}^{\ddagger}$  on  $V_y$ ; here  $\eta_y^0 : V_y \rightarrow Z_s^z$  is the restriction of  $\eta_y : V'_y \rightarrow \bar{Z}_s^z$ . By 3.9(a),

$$j^* (p_{01}^* L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^* L_{\lambda}^{\ddagger\sharp} \otimes p_{23}^* L_{y(\lambda)}^{\dot{y}^{-1}})$$

is the intersection cohomology complex of  $V'_y$  with coefficients in the local system

$$\tilde{L} = j^* (p_{01}^* L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^* L_{\lambda}^{\ddagger} \otimes p_{23}^* L_{y(\lambda)}^{\dot{y}^{-1}})$$

on  $V_y$ . It is then enough to show that  $\tilde{L} = (\eta_y^0)^* \mathcal{L}_{\lambda,s}^{\ddagger}$ .

Let  $\xi = (x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}) \in V_y$ . From the definition of  $\eta_y^0$  we see that the stalk  $((\eta_y^0)^* \mathcal{L}_{\lambda,s}^{\ddagger})_{\xi}$  is equal to

$$(\mathcal{L}_{\lambda,s}^{\ddagger})_{\epsilon_s(x_1 t_1^{-1}, x_2 t_2^{-1})} = (L_{\lambda})_{t_0}$$

where  $t_0 \in \mathbf{T}$ ,  $t_1 \in \mathbf{T}$ ,  $t_2 \in \mathbf{T}$  are such that  $t_0 = (t_1 x_1^{-1} x_2 t_2^{-1})_{\dot{z}}$ ,

$$\mathbf{e}^s((x_0^{-1} x_1 t_1^{-1})_{\mathbf{e}^{-s}(\dot{y})}) = (x_3^{-1} x_2 t_2^{-1})_{\dot{y}},$$

We can choose  $t_1, t_2$  so that

$$(x_0^{-1} x_1 t_1^{-1})_{\mathbf{e}^{-s}(\dot{y})} = 1, (x_3^{-1} x_2 t_2^{-1})_{\dot{y}} = 1;$$

thus we can assume that  $t_1 = (x_0^{-1} x_1)_{\mathbf{e}^{-s}(\dot{y})}$ ,  $t_2 = (x_3^{-1} x_2)_{\dot{y}} = 1$ .

The stalk  $\tilde{L}_{\xi}$  is  $(L_{z(\lambda)})_{t'_1} \otimes (L_{\lambda})_{t'_2} \otimes (L_{y(\lambda)})_{t'_3}$  where

$$t'_1 = (x_0^{-1} x_1)_{\mathbf{e}^{-s}(\dot{y})} \in \mathbf{T}, t'_2 = (x_1^{-1} x_2)_{\dot{z}} \in \mathbf{T}, t'_3 = (x_2^{-1} x_3)_{\dot{y}^{-1}} \in \mathbf{T}.$$

It is enough to show that  $(\eta_y^* \mathcal{L}_{\lambda,s}^{\ddagger})_{\xi} = \tilde{L}_{\xi}$ , or that

$$(t_1 x_1^{-1} x_2 t_2^{-1})_{\dot{z}} = z^{-1}(t'_1) t'_2 y^{-1}(t'_3)$$

where  $t_1, t_2, t'_1, t'_2, t'_3$  are as above. We have  $t_1 = t'_1$  and  $x_3^{-1} x_2 \in \mathbf{U} \dot{y} t_2 \mathbf{U}$ , hence

$$x_2^{-1} x_3 \in \mathbf{U} t_2^{-1} \dot{y}^{-1} \mathbf{U} = \mathbf{U} \dot{y}^{-1} y(t_2^{-1}) \mathbf{U},$$

so that  $t'_3 = y(t_2^{-1})$  and  $t_2^{-1} = y^{-1}(t'_3)$ . We have

$$t_1 x_1^{-1} x_2 t_2^{-1} \in t_1 \mathbf{U} \dot{z} t'_2 \mathbf{U} t_2^{-1} = \mathbf{U} \dot{z} z^{-1}(t_1) t'_2 t_2^{-1} \mathbf{U},$$

so that

$$(t_1 x_1^{-1} x_2 t_2^{-1})_{\dot{z}} = z^{-1}(t_1) t'_2 t_2^{-1} = z^{-1}(t'_1) t'_2 y^{-1}(t'_3),$$

as required. This completes the proof of (b) hence that of (a).

**4.6.** Let

$$(w_1, w_2, \dots, w_f, w_{f+2}, w_{f+4}, \dots, w_r) \in W^{r-2},$$

$$(\lambda_1, \lambda_2, \dots, \lambda_f, \lambda_{f+2}, \lambda_{f+4}, \dots, \lambda_r) \in \mathfrak{s}_n^{r-2}.$$

We set  $z = w_{f+2}, \lambda = \lambda_{f+2}$ . We assume that  $z(\lambda) = \mathbf{e}^{-s}(\lambda)$ . Let  $P$  be as in 4.5. Let

$$P' = \otimes_{i \in [1, r] - \{f+1, f+2, f+3\}} p_{i-1, i}^* L_{\lambda_i}^{\dot{w}_i \#} \in \mathcal{D}_m(\tilde{\mathcal{B}}^{r+1}),$$

$\tilde{P} = P \otimes \vartheta^* P' \in \mathcal{D}_m(\mathcal{Y})$ . For any  $y \in W$  we set

$$\mathbf{w}_y = (w_1, w_2, \dots, w_f, \mathbf{e}^{-s}(y), w_{f+2}, y^{-1}, w_{f+4}, \dots, w_r) \in W^r,$$

$$\omega_y = (\dot{w}_1, \dot{w}_2, \dots, \dot{w}_f, \mathbf{e}^{-s}(\dot{y}), \dot{w}_{f+2}, \dot{y}^{-1}, \dot{w}_{f+4}, \dots, \dot{w}_r),$$

$$\lambda_y = (\lambda_1, \lambda_2, \dots, \lambda_f, \mathbf{e}^{-s}(\lambda_{f+2}), \lambda_{f+2}, y(\lambda_{f+2}), \lambda_{f+4}, \dots, \lambda_r) \in \mathfrak{s}_n^r.$$

We set  $\Xi = \vartheta_! \tilde{P}$ . We have:

$$(a) \quad \Xi \simeq \{M_{\lambda_y}^{\omega_y, [1, r] - \{f+1, f+3\}} \langle 2|y| - 2\nu \rangle; y \in W\}$$

in  $\mathcal{D}_m(\tilde{\mathcal{B}}^{r+1})$ . This follows immediately from 4.5(a) since  $\Xi = P' \otimes \vartheta_!(P)$ .

**4.7.** We preserve the setup of 4.6. Let  $\mathcal{S} = \sqcup_{\mathbf{w}'} \tilde{\mathcal{O}}_{\mathbf{w}'}^\emptyset$ , where the union is over all  $\mathbf{w}' = (w'_1, \dots, w'_r) \in W^r$  such that  $w'_i = w_i$  for  $i \notin \{f+1, f+3\}$ . This is a locally closed subvariety of  $\tilde{\mathcal{B}}^{r+1}$ . For  $y \in W$  let  $R_y$  be the restriction of  $M_{\lambda_y}^{\omega_y, \emptyset}$  to  $\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset$  extended by 0 on  $\mathcal{S} - \tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset$  (a constructible sheaf on  $\mathcal{S}$ ). From the definitions we have

$$M_{\lambda_y}^{\omega_y, [1, r] - \{f+1, f+3\}}|_{\mathcal{S}} = R_y.$$

From 4.6(a) we deduce  $\Xi|_{\mathcal{S}} \simeq \{R_y \langle 2|y| - 2\nu \rangle; y \in W\}$ . We now restrict further to  $\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset$  (for  $y \in W$ ); we obtain

$$\Xi|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset} \simeq \{R_{y'} \langle 2|y'| - 2\nu \rangle|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset}; y' \in W\}.$$

In the right hand side we have  $R_{y'} \langle 2|y'| - 2\nu \rangle|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset} = 0$  if  $y' \neq y$ . It follows that  $\Xi|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset} = R_y \langle 2|y| - 2\nu \rangle|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset}$ . Since  $R_y|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset}$  is a local system we deduce for  $y \in W$  the following result.

(a) *Let  $h \in \mathbf{Z}$ . If  $h = 2\nu - 2|y|$  then  $\mathcal{H}^h \Xi|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset} = R_y|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset}(|y| - \nu)$ . If  $h \neq 2\nu - 2|y|$ , then  $\mathcal{H}^h \Xi|_{\tilde{\mathcal{O}}_{\mathbf{w}_y}^\emptyset} = 0$ .*

**4.8.** We preserve the setup of 4.6. We set

$$(a) \quad k = 3\nu + (r+1)\rho + \sum_{i \in [1,r] - \{f+1, f+3\}} |w_i|.$$

For  $y \in W$  we set

$$K_y = M_{\lambda_y}^{\omega_y, [1,r] - \{f+1, f+3\}} \langle |\mathbf{w}_y| + \nu + (r+1)\rho \rangle,$$

$$\tilde{K}_y = M_{\lambda_y}^{\omega_y, [1,r]} \langle |\mathbf{w}_y| + \nu + (r+1)\rho \rangle.$$

From 4.6(a) we deduce:

$$(b) \quad \Xi\langle k \rangle \simeq \{K_y; y \in W\}.$$

We show:

(c) *For any  $j > 0$  we have  $(\Xi\langle k \rangle)^j = 0$ . Equivalently,  $\Xi^j = 0$  for any  $j > k$ .*

Using (b) we see that it is enough to show that for any  $y \in W$  we have  $(K_y)^j = 0$  for any  $j > 0$ . Now  $\tilde{K}_y$  is a (simple) perverse sheaf hence for any  $j$  we have  $\dim \text{supp} \mathcal{H}^j \tilde{K}_y \leq -j$ . Moreover  $K_y$  is obtained by restricting  $\tilde{K}_y$  to an open subset of its support and then extending the result (by zero) on the complement of this subset in  $\tilde{\mathcal{B}}^{r+1}$ . Hence  $\text{supp} \mathcal{H}^j K_y \subset \text{supp} \mathcal{H}^j \tilde{K}_y$  so that  $\dim \text{supp} \mathcal{H}^j K_y \leq -j$ . Since this holds for any  $j$  we see that  $(K_y)^j = 0$  for any  $j > 0$ .

**4.9.** We preserve the notation of 4.6. We show:

(a) *Let  $j \in \mathbf{Z}$  and let  $X$  be a composition factor of  $\Xi^j$ . Then  $X \cong M_{\lambda'}^{\omega', [1,r]} \langle |\mathbf{w}'| + \nu + (r+1)\rho \rangle$  for some*

$$\mathbf{w}' = (w'_1, w'_2, \dots, w'_r) \in W^r, \lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_r) \in \mathfrak{s}_n^r$$

*such that  $w'_i = w_i$ ,  $\lambda'_i = \lambda_i$  for  $i \in [1, r] - \{f+1, f+3\}$  and such that*

$$\lambda'_{f+1} = w'_{f+2}(\lambda'_{f+2}), \lambda'_{f+2} = w'_{f+3}(\lambda'_{f+3}).$$

*Here  $\omega' = (w'_1, w'_2, \dots, w'_r)$ .*

From 4.6(a) we see that, for some  $y \in W$ ,  $X$  is a composition factor of

$$(M_{\lambda_y}^{\omega_y, [1,r] - \{f+1, f+3\}} \langle 2|y| - 2\nu \rangle)^j$$

where  $\omega_y, \lambda_y$  are as in 4.6. Using this and [L16, 2.18(b)] we see that

$$X \cong M_{\lambda'}^{\omega', [1,r]} \langle |\mathbf{w}'| + \nu + (r+1)\rho \rangle$$

for some

$$\mathbf{w}' = (w'_1, w'_2, \dots, w'_r) \in W^r, \lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_r) \in \mathfrak{s}_n^r$$



such that  $w'_i = w_i$ ,  $\lambda'_i = \lambda_i$  for  $i \in [1, r] - \{f+1, f+3\}$ ; here  $\omega' = (w'_1, w'_2, \dots, w'_r)$ . It remains to show that we have automatically

$$\lambda'_{f+1} = w'_{f+2}(\lambda'_{f+2}), \lambda'_{f+2} = w'_{f+3}(\lambda'_{f+3}).$$

To see this we note that  $(M_{\lambda_y}^{\omega_y, [1, r] - \{f+1, f+3\}} \langle 2|y| - 2\nu \rangle)^j$  is equivariant for the  $\mathbf{T}^2$ -action

$$(t_1, t_2) : (x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_r \mathbf{U}) \mapsto (x_0 \mathbf{U}, x_1 \mathbf{U}, \dots, x_f \mathbf{U}, x_{f+1} t_1 \mathbf{U}, x_{f+2} t_2 \mathbf{U}, x_{f+3} \mathbf{U}, \dots, x_r \mathbf{U})$$

hence so are its composition factors and this implies that the equalities above for  $\lambda'_{f+1}, \lambda'_{f+2}$  do hold.

**4.10.** From 4.8(c) we see that we have a distinguished triangle  $(\Xi', \Xi, \Xi^k[-k])$  where  $\Xi' \in \mathcal{D}_m(\tilde{\mathcal{B}}^{r+1})$  satisfies  $(\Xi')^j = 0$  for all  $j \geq k$ . We show:

(a) *Let  $j \in \mathbf{Z}$  and let  $K$  be one of  $\Xi, \Xi^j, \Xi'$ . For any  $\mathbf{w}' \in W^r$  and any  $h \in \mathbf{Z}$ ,  $\mathcal{H}^h K|_{\tilde{\mathcal{O}}_{\mathbf{w}'}}^{\emptyset}$  is a local system.*

We prove (a) for  $K = \Xi$  or  $K = \Xi^j$ . Using 4.6(a), we see that it is enough to show that  $\mathcal{H}^h(M_{\lambda_y}^{\omega_y, [1, r] - \{f+1, f+3\}})|_{\tilde{\mathcal{O}}_{\mathbf{w}'}}^{\emptyset}$  is a local system for any  $h$  and that  $\mathcal{H}^h((M_{\lambda_y}^{\omega_y, [1, r] - \{f+1, f+3\}})^j)|_{\tilde{\mathcal{O}}_{\mathbf{w}'}}^{\emptyset}$  is a local system for any  $h$  and any  $j$ . This follows by an argument entirely similar to that in the proof of [L16, 3.10].

Now (a) for  $K = \Xi'$  follows from (a) for  $\Xi$  and  $\Xi^k[-k]$  using the long exact sequence for cohomology sheaves of  $(\Xi', \Xi, \Xi^k[-k])$  restricted to  $\tilde{\mathcal{O}}_{\mathbf{w}'}^{\emptyset}$ .

We show:

(b) *Let  $(y', y'') \in W^2$ ,  $j = 2\nu - |y'| - |y''|$ . Let*

$$\mathbf{w}_{y', y''} = (w_1, w_2, \dots, w_f, y', w_{f+2}, y''^{-1}, w_{f+3}, \dots, w_r) \in W^r.$$

*The induced homomorphism  $\mathcal{H}^j \Xi|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}} \rightarrow \mathcal{H}^{j-k}(\Xi^k)|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}}$  is an isomorphism.*

We have an exact sequence of constructible sheaves

$$\mathcal{H}^j \Xi'|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}} \rightarrow \mathcal{H}^j \Xi|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}} \rightarrow \mathcal{H}^{j-k}(\Xi^k)|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}} \rightarrow \mathcal{H}^{j+1} \Xi'|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}}.$$

Hence it is enough to show that  $\mathcal{H}^{j'} \Xi'|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}} = 0$  if  $j' \geq j$ . Assume that  $\mathcal{H}^{j'} \Xi'|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}} \neq 0$  for some  $j' \geq j$ . Since  $\mathcal{H}^{j'} \Xi'|_{\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}}$  is a local system (see (a)), we deduce that  $\tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset}$  is contained in  $\text{supp}(\mathcal{H}^{j'} \Xi')$ . We have  $(\Xi'[k-1])^{\tilde{j}} = 0$  for any  $\tilde{j} > 0$  hence  $\dim \text{supp}(\mathcal{H}^{j''} \Xi'[k-1]) \leq -j''$  for any  $j''$ . Taking  $j'' = j' - k + 1$ , we deduce that

$$\dim \tilde{\mathcal{O}}_{\mathbf{w}_{y', y''}}^{\emptyset} \leq \dim \text{supp}(\mathcal{H}^{j'} \Xi') \leq -j' + k - 1 \leq -j + k - 1$$

hence

$$|\mathbf{w}_{y', y''}| + \nu + (r+1)\rho \leq -j + k - 1.$$

We have  $|\mathbf{w}_{y', y''}| + \nu + (r+1)\rho = -j + k$  hence  $-j + k \leq -j + k - 1$ , contradiction. This proves (b).

**4.11.** For  $(y', y'') \in W^2$  we set

$$\begin{aligned}\omega_{y', y''} &= (\dot{w}_1, \dot{w}_2, \dots, \dot{w}_f, \dot{y}', \dot{w}_{f+2}, \dot{y}''^{-1}, \dot{w}_{f+3}, \dots, \dot{w}_r) \in W^r, \\ \lambda_{y', y''} &= (\lambda_1, \lambda_2, \dots, \lambda_f, \mathbf{e}^{-s}(\lambda_{f+2}), \lambda_{f+2}, y''(\lambda_{f+2}), \lambda_{f+4}, \dots, \lambda_r) \in \mathfrak{s}_n^r, \\ K_{y', y''} &= M_{\lambda_{y', y''}}^{\omega_{y', y''}, \emptyset} \langle |\mathbf{w}_{y', y''}| + \nu + (r+1)\rho \rangle \in \mathcal{M}_m(\tilde{\mathcal{B}}^{r+1}), \\ \tilde{K}_{y', y''} &= M_{\lambda_{y', y''}}^{\omega_{y', y''}, [1, r]} \langle |\mathbf{w}_{y', y''}| + \nu + (r+1)\rho \rangle \in \mathcal{M}_m(\tilde{\mathcal{B}}^{r+1}).\end{aligned}$$

Note that when  $y' = \mathbf{e}^{-s}(y)$ ,  $y'' = y$ ,  $\mathbf{w}_{y', y''}, \omega_{y', y''}, \lambda_{y', y''}$  and  $\tilde{K}_{y', y''}$  become  $\mathbf{w}_y, \omega_y, \lambda_y$  (see 4.6) and  $\tilde{K}_y$  (see 4.8). We show that we have canonically

$$(a) \quad gr_0(\Xi^k(k/2)) = \oplus_{y \in W} \tilde{K}_y.$$

Since  $gr_0(\Xi^k(k/2))$  is a semisimple perverse sheaf of pure weight zero, it is a direct sum of simple perverse sheaves, necessarily of the form described in 4.9(a). Thus we have canonically

$$gr_0(\Xi^k(k/2)) = \oplus_{(y', y'') \in W^2} V_{y', y''} \otimes \tilde{K}_{y', y''}$$

where  $V_{y', y''}$  are mixed  $\bar{\mathbf{Q}}_l$ -vector spaces of pure weight 0. By [BBD, 5.1.14],  $\Xi$  is mixed of weight  $\leq 0$  hence  $\Xi^k(k/2)$  is mixed of weight  $\leq 0$ . Hence we have an exact sequence in  $\mathcal{M}_m(\tilde{\mathcal{B}}^{r+1})$ :

$$(a) \quad 0 \rightarrow \mathcal{W}^{-1}(\Xi^k(k/2)) \rightarrow \Xi^k(k/2) \rightarrow gr_0(\Xi^k(k/2)) \rightarrow 0$$

that is,

$$0 \rightarrow \mathcal{W}^{-1}(\Xi^k(k/2)) \rightarrow \Xi^k(k/2) \rightarrow \oplus_{(y', y'') \in W^2} V_{y', y''} \otimes \tilde{K}_{y', y''} \rightarrow 0.$$

(Here  $\mathcal{W}^{-1}(?)$  denotes the part of weight  $\leq -1$  of a mixed perverse sheaf.) Hence for any  $(\tilde{y}', \tilde{y}'') \in W^2$  we have an exact sequence of (mixed) cohomology sheaves restricted to  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^{\emptyset}$  (where  $h = 2\nu - |\tilde{y}'| - |\tilde{y}''| - k$ ):

$$\begin{aligned}(b) \quad & \mathcal{H}^h(\mathcal{W}^{-1}(\Xi^k(k/2))) \xrightarrow{\alpha} \mathcal{H}^h(\Xi^k(k/2)) \rightarrow \oplus_{(y', y'') \in W^2} V_{y', y''} \otimes \mathcal{H}^h(\tilde{K}_{y', y''}) \rightarrow \\ & \mathcal{H}^{h+1}(\mathcal{W}^{-1}(\Xi^k(k/2))).\end{aligned}$$

Moreover, by 4.10(b), we have an equality of local systems on  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^{\emptyset}$ :

$$\mathcal{H}^h(\Xi^k(k/2)) = \mathcal{H}^{h+k}(\Xi(k/2)) = \mathcal{H}^{2\nu - |\tilde{y}'| - |\tilde{y}''|}(\Xi(k/2))$$

and this is  $R_y(k/2 + |y| - \nu)$  if  $\tilde{y}' = \mathbf{e}^{-s}(y)$ ,  $\tilde{y}'' = y$  (see 4.7(a)) and is 0 if  $\tilde{y}' \neq \mathbf{e}^{-s}(\tilde{y}'')$  (see 4.4(a)) hence is pure of weight  $-k - |\tilde{y}'| - |\tilde{y}''| + \nu = h$ . On the other hand,  $\mathcal{H}^h(\mathcal{W}^{-1}(\Xi^k(k/2)))$  is mixed of weight  $\leq h - 1$ ; it follows that  $\alpha$  in (b) must be zero.

Assume that  $\mathcal{H}^h(\tilde{K}_{y', y''})$  is not identically zero on  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset$ . Then, by 4.10(a),  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset$  is contained in  $\text{supp} \mathcal{H}^h(\tilde{K}_{y', y''})$  which has dimension  $\leq -h$  (resp.  $< -h$  if  $(y', y'') \neq (\tilde{y}', \tilde{y}'')$ ); hence  $-h = \dim \tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset$  is  $\leq -h$  (resp.  $< -h$ ); we see that we must have  $(y', y'') = (\tilde{y}', \tilde{y}'')$  and we have  $\mathcal{H}^h(\tilde{K}_{y', y''}) = \mathcal{H}^h(K_{y', y''})$  on  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset$ .

Assume that  $\mathcal{H}^{h+1}(\mathcal{W}^{-1}(\Xi^k(k/2)))$  is not identically 0 on  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset$ . Then, by 4.10(a),  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset$  is contained in  $\text{supp} \mathcal{H}^{h+1}(\mathcal{W}^{-1}(\Xi^k(k/2)))$  which has dimension  $\leq -h - 1$ ; hence  $-h = \dim \tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset \leq -h - 1$ , a contradiction. We see that (b) becomes an isomorphism of local systems on  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset$ :

$$0 = V_{\tilde{y}', \tilde{y}''} \otimes K_{\tilde{y}', \tilde{y}''} \text{ if } \mathbf{e}^s(\tilde{y}') \neq \tilde{y}'',$$

$$R_{\tilde{y}''}(-h/2) \xrightarrow{\sim} V_{\tilde{y}', \tilde{y}''} \otimes \mathcal{H}^h(K_{\tilde{y}', \tilde{y}''}) \text{ if } \mathbf{e}^s(\tilde{y}') = \tilde{y}''.$$

When  $\mathbf{e}^s(\tilde{y}') = \tilde{y}'$  we have  $\mathcal{H}^h(K_{\tilde{y}', \tilde{y}''}) = R_{\tilde{y}''}(-h/2)$  as local systems on  $\tilde{\mathcal{O}}_{\mathbf{w}_{\tilde{y}', \tilde{y}''}}^\emptyset$ . It follows that  $V_{\tilde{y}', \tilde{y}''}$  is  $\bar{\mathbf{Q}}_l$  if  $\mathbf{e}^s(\tilde{y}') = \tilde{y}''$  and is 0 if  $\mathbf{e}^s(\tilde{y}') \neq \tilde{y}''$ . This proves (a).

**4.12.** Let  $h \in [1, r]$ . Let  ${}_h\mathcal{D}^{\preceq} \tilde{\mathcal{B}}^{r+1}$  (resp.  ${}_h\mathcal{D}^{\prec} \tilde{\mathcal{B}}^{r+1}$ ) be the subcategory of  $\mathcal{D} \tilde{\mathcal{B}}^{r+1}$  consisting of objects  $K$  such that for any  $j \in \mathbf{Z}$ , any composition factor of  $K^j$  is of the form  $M_{\lambda}^{\omega, [1, r]} \langle |\mathbf{w}| + \nu + (r+1)\rho \rangle$  for some  $\mathbf{w} = (w_1, \dots, w_r) \in W^r$ ,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathfrak{s}_n^r$  such that  $w_h \cdot \lambda_h \preceq \mathbf{c}$  (resp.  $w_h \cdot \lambda_h \prec \mathbf{c}$ ). (Here  $\omega = (\dot{w}_1, \dot{w}_2, \dots, \dot{w}_r)$ .)

Let  ${}_h\mathcal{M}^{\preceq} \tilde{\mathcal{B}}^{r+1}$  be the subcategory of  ${}_h\mathcal{D}^{\preceq} \tilde{\mathcal{B}}^{r+1}$  consisting of perverse sheaves. Let  ${}_h\mathcal{M}^{\prec} \tilde{\mathcal{B}}^{r+1}$  be the subcategory of  ${}_h\mathcal{D}^{\prec} \tilde{\mathcal{B}}^{r+1}$  consisting of perverse sheaves.

If  $K \in \mathcal{M}_m(\tilde{\mathcal{B}}^{r+1})$  is pure of weight 0 and is also in  ${}_h\mathcal{D}^{\preceq} \tilde{\mathcal{B}}^{r+1}$ , we denote by  $\underline{K}$  the sum of all simple subobjects of  $K$  (without mixed structure) which are not in  ${}_h\mathcal{D}^{\prec} \tilde{\mathcal{B}}^{r+1}$ .

**4.13.** Let  $Z_s \xleftarrow{\eta} \mathcal{Y} \xrightarrow{\vartheta} \tilde{\mathcal{B}}^4$  be as in 4.4 with  $r = 3, f = 0$ . We define  $\mathfrak{b} : \mathcal{D}(Z_s) \rightarrow \mathcal{D}(\tilde{\mathcal{B}}^2)$  and  $\mathfrak{b} : \mathcal{D}_m(Z_s) \rightarrow \mathcal{D}_m(\tilde{\mathcal{B}}^2)$  by

$$\mathfrak{b}(L) = p_{03!} \vartheta_! \eta^* L.$$

We show:

- (a) If  $L \in \mathcal{D}^{\preceq}(Z_s)$  then  $\mathfrak{b}(L) \in \mathcal{D}^{\preceq} \tilde{\mathcal{B}}^2$ .
- (b) If  $L \in \mathcal{D}^{\prec}(Z_s)$  then  $\mathfrak{b}(L) \in \mathcal{D}^{\prec} \tilde{\mathcal{B}}^2$ .
- (c) If  $L \in \mathcal{M}^{\preceq}(Z_s)$  and  $h > 5\rho + 2\nu + 2a$  then  $(\mathfrak{b}(L))^h \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ .

We can assume that  $L = \mathbb{L}_{\lambda, s}^{\dot{z}}$  where  $z \cdot \lambda \in I_n^s$ ,  $z \cdot \lambda \preceq \mathbf{c}$ . Applying 4.5(a) with  $P = \eta^* \mathcal{L}_{\lambda, s}^{\dot{z}^\#}$  we see that

$$\mathfrak{b}(\mathcal{L}_{\lambda, s}^{\dot{z}^\#}) \simeq \{L_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, \{2\}} \langle -|z| - 2\nu \rangle; y \in W\},$$

hence

$$\mathbf{b}(\mathbb{L}_{\lambda,s}^{\dot{z}}) \simeq \{L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}} \langle -\nu + \rho \rangle; y \in W\}.$$

To prove (a) it is enough to show that for any  $y \in W$  we have

$$L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}} \in \mathcal{D}^{\preceq} \tilde{\mathcal{B}}^2.$$

When  $z \cdot \lambda \in \mathbf{c}$  this follows from [L16, 2.10(a)]. When  $z \cdot \lambda \prec \mathbf{c}$  this again follows from [L16, 2.10(a)], applied to the two-sided cell containing  $z \cdot \lambda$  instead of  $\mathbf{c}$ . The same argument proves (b). To prove (c) we can assume that  $z \cdot \lambda \in \mathbf{c}$ ; it is enough to prove that for any  $y \in W$  we have

$$(L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}} \langle -\nu + \rho \rangle)^h \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$$

if  $h > 5\rho + 2\nu + 2a$  or that

$$(L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}})^j \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$$

if  $j > 6\rho + \nu + 2a$ . This follows from [L16, 2.20(a)]. This completes the proof of (a),(b),(c).

We define  $\underline{\mathbf{b}} : \mathcal{C}_0^{\mathbf{c}}(Z_s) \rightarrow \mathcal{C}_0^{\mathbf{c}}(\tilde{\mathcal{B}}^2)$  by

$$\underline{\mathbf{b}}(L) = \underline{gr_{5\rho+2\nu+2a}((\mathbf{b}(L))^{5\rho+2\nu+2a})((5\rho+2\nu+2a)/2)}.$$

We show:

(d) *Let  $z \cdot \lambda \in \mathbf{c}^s$ . If  $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$ , then*

$$\underline{\mathbf{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = \oplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \circ \mathbf{L}_{\lambda}^{\dot{z}} \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}.$$

*If  $\mathbf{e}^s(\mathbf{c}) \neq \mathbf{c}$ , then  $\underline{\mathbf{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = 0$ .*

We shall apply the method of [L14, 1.12] with  $\Phi : \mathcal{D}_m(Y_1) \rightarrow \mathcal{D}_m(Y_2)$  replaced by  $p_{03!} : \mathcal{D}_m(\tilde{\mathcal{B}}^4) \rightarrow \mathcal{D}_m(\tilde{\mathcal{B}}^2)$  and with  $\mathcal{D}^{\preceq}(Y_1)$ ,  $\mathcal{D}^{\preceq}(Y_2)$  replaced by  ${}_2\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^2)$ ,  ${}_2\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$ , see 4.12. We shall take  $\mathbf{X}$  in *loc.cit.* equal to  $\vartheta_! \eta^* \mathbb{L}_{\lambda,s}^{\dot{z}}$ . The conditions of *loc.cit.* are satisfied: those concerning  $\mathbf{X}$  are satisfied with  $c' = 2\nu + 3\rho$ . (For  $h > |z| + 3\nu + 4\rho$  we have  $\Xi^h = 0$  that is  $(\mathbf{X}[-|z| - \nu - \rho])^h = 0$ , with  $\Xi$  as in 4.8(c). Hence if  $j > 2\nu + 3\rho$  we have  $\mathbf{X}^j = 0$ .) The conditions concerning  $p_{03!}$  are satisfied with  $c = 2\rho + 2a$ . (This follows from [L16, 2.20(a)].) Since  $\mathbf{b}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = p_{03!} \mathbf{X}$  and  $c + c' = 5\rho + 2\nu + 2a$ , we see that

$$\underline{\mathbf{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = \underline{gr_{2\rho+2a}(p_{03!}((gr_{2\nu+3\rho}((\vartheta_! \eta^* \mathbb{L}_{\lambda,s}^{\dot{z}})^{2\nu+3\rho})((2\nu+3\rho)/2)))^{2\rho+2a})(\rho+a)}.$$

Using 4.11(a), we see that (with  $\Xi$  as in 4.11(a) and  $k = |z| + 3\nu + 4\rho$ ) we have

$$\begin{aligned} & \frac{gr_{2\nu+3\rho}((\vartheta_! \eta^* \mathbb{L}_{\lambda,s}^{\dot{z}})^{2\nu+3\rho})((2\nu+3\rho)/2)}{=} \\ &= \frac{gr_{2\nu+3\rho}((\Xi(|z| + \nu + \rho))^{2\nu+3\rho})((2\nu+3\rho)/2)}{=} \\ &= \underline{gr_0(\Xi^k(k/2))} = \oplus_{y \in W} M_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, [1,3]} \langle 2|y| + |z| + \nu + 4\rho \rangle. \end{aligned}$$

Hence

$$\begin{aligned} \underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}}) &= \underline{gr_{2\rho+2a}(\oplus_{y \in W} (p_{03}! M_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, [1,3]} \langle 2|y| + |z| + \nu + 4\rho \rangle)^{2\rho+2a})}(\rho + a) \\ &= \underline{gr_{2\rho+2a}(\oplus_{y \in W} (L_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, [1,3]})^{6\rho+\nu+2a}((\nu+4\rho)/2))}(\rho + a). \end{aligned}$$

Using [L16, 2.26(a)], we see that in the last direct sum, the contribution of  $y \in W$  is 0 unless  $y \cdot \lambda \in \mathbf{c}$  and  $\mathbf{e}^{-s}(y) \cdot \mathbf{e}^{-s}(\lambda) \in \mathbf{c}$ . We see that the last direct sum is zero unless  $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$ . If  $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$ , for the terms corresponding to  $y$  such that  $y \cdot \lambda \in \mathbf{c}$ , we may apply [L16, 2.24(a)]. Now (d) follows.

**4.14.** We set  $\mathbf{Z}_{\mathbf{c}} = \{s' \in \mathbf{Z}; \mathbf{e}^{s'}(\mathbf{c}) = \mathbf{c}\}$ . This is a subgroup of  $\mathbf{Z}$ . In the remainder of this section we assume that  $s \in \mathbf{Z}_{\mathbf{c}}$ .

Let  $Z_s \xleftarrow{! \eta} !\mathcal{Y}$  be as in 4.4 with  $r = 3, f = 0$ . Let  ${}^! \tilde{\mathcal{B}}^4$  be the space of orbits of the free  $\mathbf{T}^2$ -action on  $\tilde{\mathcal{B}}^4$  given by

$$(t_1, t_2) : (x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}) \mapsto (x_0 \mathbf{U}, x_1 t_1 \mathbf{U}, x_2 t_2 \mathbf{U}, x_3 \mathbf{U});$$

let  ${}^! \vartheta : {}^! \mathcal{Y} \rightarrow {}^! \tilde{\mathcal{B}}^4$  be the map induced by  $\vartheta$ . We define  $\mathfrak{b}' : \mathcal{D}(Z_s) \rightarrow \mathcal{D}(\tilde{\mathcal{B}}^2)$  and  $\mathfrak{b}' : \mathcal{D}_m(Z_s) \rightarrow \mathcal{D}_m(\tilde{\mathcal{B}}^2)$  by

$$\mathfrak{b}'(L) = p_{03}! {}^! \vartheta_! {}^! \eta^* L.$$

(The map  ${}^! \tilde{\mathcal{B}}^4 \rightarrow \tilde{\mathcal{B}}^2$  induced by  $p_{03} : \tilde{\mathcal{B}}^4 \rightarrow \tilde{\mathcal{B}}^2$  is denoted again by  $p_{03}$ .) Let  $\tau : \mathcal{Y} \rightarrow {}^! \mathcal{Y}$  be as in 4.4 (it is a principal  $\mathbf{T}^2$ -bundle). We have the following results.

- (a) If  $L \in \mathcal{D}^{\preceq}(Z_s)$ , then  $\mathfrak{b}'(L) \in \mathcal{D}^{\preceq} \tilde{\mathcal{B}}^2$ .
- (b) If  $L \in \mathcal{D}^{\prec}(Z_s)$ , then  $\mathfrak{b}'(L) \in \mathcal{D}^{\prec} \tilde{\mathcal{B}}^2$ .
- (c) If  $L \in \mathcal{M}^{\preceq}(Z_s)$  and  $h > \rho + 2\nu + 2a$ , then  $(\mathfrak{b}'(L))^h \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ .

We can assume that  $L = \mathbb{L}_{\lambda,s}^{\dot{z}}$  where  $z \cdot \lambda \in I_n^s$ ,  $z \cdot \lambda \preceq \mathbf{c}$ . A variant of the proof of 4.5(a) gives:

$$\mathfrak{b}'(\mathcal{L}_{\lambda,s}^{\dot{z}\sharp}) \simeq \{L_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, \{2\}} \langle -|z| - 2\nu \rangle; y \in W\},$$

hence

$$\mathfrak{b}'(\mathbb{L}_{\lambda,s}^{\dot{z}\sharp}) \simeq \{L_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, \{2\}} \langle -\nu + \rho \rangle; y \in W\}.$$

To prove (a) it is enough to show that for any  $y \in W$  we have

$$'L_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, \{2\}} \in \mathcal{D}^{\preceq} \tilde{\mathcal{B}}^2.$$

When  $z \cdot \lambda \in \mathbf{c}$  this follows from [L16, 2.10(c)]. When  $z \cdot \lambda \prec \mathbf{c}$  this again follows from [L16, 2.10(c)], applied to the two-sided cell containing  $z \cdot \lambda$  instead of  $\mathbf{c}$ . The same argument proves (b). To prove (c) we can assume that  $z \cdot \lambda \in \mathbf{c}$ ; it is enough to prove that for any  $y \in W$  we have

$$('L_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, \{2\}} \langle -\nu + \rho \rangle)^h \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$$

if  $h > \rho + 2\nu + 2a$  or that  $('L_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{z}, \dot{y}^{-1}, \{2\}})^j \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$  if  $j > 2\rho + \nu + 2a$ . This follows from [L16, 2.20(c)]. This completes the proof of (a),(b),(c).

We define  $\underline{\mathbf{b}}' : \mathcal{C}_0^{\mathbf{c}}(Z_s) \rightarrow \mathcal{C}_0^{\mathbf{c}}(\tilde{\mathcal{B}}^2)$  by

$$\underline{\mathbf{b}}'(L) = \underline{gr_{\rho+2\nu+2a}((\mathbf{b}'(L))^{\rho+2\nu+2a})((\rho+2\nu+2a)/2)}.$$

In the remainder of this subsection we fix  $z \cdot \lambda \in \mathbf{c}^s$  and we set  $L = \mathbb{L}_{\lambda, s}^{\dot{z}}$ . We show:

(d) *We have canonically  $\underline{\mathbf{b}}'(L) = \underline{\mathbf{b}}(L)$ .*

The method of proof is similar to that of [L16, 2.22(a)]. It is based on the fact that

$$\mathbf{b}(L) = \mathbf{b}'(L) \otimes \mathfrak{L}^{\otimes 2}$$

which follows from the definitions. We define  $\mathcal{R}_{i,j}$  for  $i \in [0, 2\rho + 1]$  and  $\mathcal{P}_{i,j}$  for  $i \in [0, 2\rho]$  as in [L16, 2.17], but replacing  $L^J, 'L^J, r, \delta$  by  $\mathbf{b}(L), \mathbf{b}'(L), 3, 2\rho$ . In particular, we have

$$\mathcal{P}_{i,j} = \mathcal{X}_{4\rho-i}(i-2\rho) \otimes (\mathbf{b}'(L))^{-4\rho+i+j} \text{ for } i \in [0, 2\rho]$$

where  $\mathcal{X}_{4\rho-i}$  is a free abelian group of rank  $\binom{2\rho}{i}$  and  $\mathcal{X}_{4\rho} = \mathbf{Z}$ . We have for any  $j$  an exact sequence analogous to [L16, 2.17(a)]:

$$(e) \quad \dots \rightarrow \mathcal{P}_{i,j-1} \rightarrow \mathcal{R}_{i+1,j} \rightarrow \mathcal{R}_{i,j} \rightarrow \mathcal{P}_{i,j} \rightarrow \mathcal{R}_{i+1,j+1} \rightarrow \mathcal{R}_{i,j+1} \rightarrow \dots,$$

and we have

$$\mathcal{R}_{0,j} = (\mathbf{b}(L))^j, \quad \mathcal{P}_{0,j} = (\mathbf{b}'(L))^{j-4\rho}(-2\rho).$$

We show:

(f) *If  $i \in [0, 2\rho + 1]$  then  $\mathcal{R}_{i,j} \in \mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$ .*

(g) *If  $i \in [0, 2\rho + 1]$ ,  $j > 6\rho - i + \nu + 2a$  then  $\mathcal{R}_{i,j} \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ .*

We prove (f),(g) by descending induction on  $i$  as in [L16, 2.21]. If  $i = 2\rho + 1$  then, since  $\mathcal{R}_{2\rho+1,j} = 0$ , there is nothing to prove. Now assume that  $i \in [0, 2\rho]$ . Assume that  $\lambda' \cdot w$  is such that  $\mathbf{L}_{\lambda'}^{\dot{w}}$  is a composition factor of  $\mathcal{R}_{i,j}$  (without the mixed structure). We must show that  $w \cdot \lambda' \preceq \mathbf{c}$  and that, if  $j > 6\rho - i + \nu + 2a$ , then

$w \cdot \lambda' \prec \mathbf{c}$ . Using (e), we see that  $\mathbf{L}_{\lambda'}^{\dot{w}}$  is a composition factor of  $\mathcal{R}_{i+1,j}$  or of  $\mathcal{P}_{i,j}$ . In the first case, using the induction hypothesis we see that  $w \cdot \lambda' \preceq \mathbf{c}$  and that, if  $j > 6\rho - i + \nu + 2a$  (so that  $j > 6\rho - i - 1 + \nu + 2a$ ), then  $w \cdot \lambda' \prec \mathbf{c}$ . In the second case,  $\mathbf{L}_{\lambda'}^{\dot{w}}$  is a composition factor of  $(\mathbf{b}'(L))^{-4\rho+i+j}$ . Using (a),(c), we see that  $w \cdot \lambda' \preceq \mathbf{c}$  and that, if  $j > 6\rho - i + \nu + 2a$  (so that  $-4\rho + i + j > \nu + 2\rho + 2a$ ), then  $w \cdot \lambda' \prec \mathbf{c}$ . This proves (f),(g).

We show:

(h) *Assume that  $i \in [0, 2\rho + 1]$ . Then  $\mathcal{R}_{i,j}$  is mixed of weight  $\leq j - i$ .*

We argue as in [L16, 2.22] by descending induction on  $i$ . If  $i = 2\rho + 1$  there is nothing to prove. Assume now that  $i \leq 2\rho$ . By Deligne's theorem,  $\mathbf{b}'(L)$  is mixed of weight  $\leq 0$ ; hence  $(\mathbf{b}'(L))^{-4\rho+i+j}$  is mixed of weight  $\leq -4\rho + i + j$  and  $\mathcal{X}_{4\rho-i}(i-2\rho) \otimes (\mathbf{b}'(L))^{-4\rho+i+j}$  is mixed of weight  $\leq -4\rho + i + j - 2(i-2\rho) = j - i$ . In other words,  $\mathcal{P}_{i,j}$  is mixed of weight  $\leq j - i$ . Thus in the exact sequence  $\mathcal{R}_{i+1,j} \rightarrow \mathcal{R}_{i,j} \rightarrow \mathcal{P}_{i,j}$  coming from (e) in which  $\mathcal{R}_{i+1,j}$  is mixed of weight  $\leq j - i - 1 < j - i$  (by the induction hypothesis) and  $\mathcal{P}_{i,j}$  is mixed of weight  $\leq j - i$ , we must have that  $\mathcal{R}_{i,j}$  is mixed of weight  $\leq j - i$ . This proves (h).

We now prove (d). From (e) we deduce an exact sequence

$$gr_j(\mathcal{R}_{1,j}) \rightarrow gr_j(\mathcal{R}_{0,j}) \rightarrow gr_j(\mathcal{P}_{0,j}) \rightarrow gr_j(\mathcal{R}_{1,j+1}).$$

By (h) we have  $gr_j(\mathcal{R}_{1,j}) = 0$ . We have  $gr_j(\mathcal{R}_{0,j}) = gr_j(\mathbf{b}(L)^j)$ ,  $gr_j(\mathcal{P}_{0,j}) = gr_j((\mathbf{b}'(L))^{-4\rho+j}(-2\rho))$ . Moreover, by (g) we have  $\mathcal{R}_{1,j+1} \in \mathcal{D}^{\prec} \tilde{\mathcal{B}}^2$  since  $j + 1 > 6\rho - 1 + \nu + 2a$ . It follows that  $gr_j(\mathcal{R}_{1,j+1}) \in \mathcal{D}^{\prec} \tilde{\mathcal{B}}^2$ . Thus the exact sequence above induces an isomorphism as in (d).

Let  $p'_{ij} : \tilde{\mathcal{B}}^3 \rightarrow \tilde{\mathcal{B}}^2$  be the projection to the  $ij$ -coordinate, where  $ij$  is 12, 23 or 13. Let

$$R = \mathbf{T} \backslash \{(x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x'_3 \mathbf{U}, \gamma) \in \tilde{\mathcal{B}}^4 \times G_s; \gamma \in x_2 \mathbf{U} \tau^s x_1^{-1}\}$$

where  $\mathbf{T}$  acts freely by

$$t : (x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x'_3 \mathbf{U}, \gamma) \mapsto (x_0 \mathbf{U}, x_1 \mathbf{e}^{-s}(t) \mathbf{U}, x_2 t \mathbf{U}, x'_3 \mathbf{U}, \gamma).$$

We have cartesian diagrams

$$\begin{array}{ccc} R & \xrightarrow{d_1} & {}'\mathcal{Y} \times \tilde{\mathcal{B}}^2 \\ c_1 \downarrow & & s_1 \downarrow \\ \tilde{\mathcal{B}}^3 & \xrightarrow{p'} & \tilde{\mathcal{B}}^2 \times \tilde{\mathcal{B}}^2 \\ \\ R & \xrightarrow{d_2} & \tilde{\mathcal{B}}^2 \times {}'\mathcal{Y} \\ c_2 \downarrow & & s_2 \downarrow \\ \tilde{\mathcal{B}}^3 & \xrightarrow{p'} & \tilde{\mathcal{B}}^2 \times \tilde{\mathcal{B}}^2 \end{array}$$

where

$$d_1(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) = ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, \gamma x_0\tau^{-s}\mathbf{U}, \gamma), (\gamma x_0\tau^{-s}\mathbf{U}, x_3\mathbf{U})).$$

$$\begin{aligned} d_2(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) \\ = ((x_0\mathbf{U}, \gamma^{-1}x_3\tau^s\mathbf{U}), (\gamma^{-1}x_3\tau^s\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma)), \end{aligned}$$

$$c_1(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) = (x_0\mathbf{U}, \gamma x_0\tau^{-s}\mathbf{U}, x_3\mathbf{U}),$$

$$c_2(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) = (x_0\mathbf{U}, \gamma^{-1}x_3\tau^s\mathbf{U}, x_3\mathbf{U}),$$

$$p' = (p'_{12}, p'_{23}), s_1 = p_{03}'\vartheta \times 1, s_2 = 1 \times p_{03}'\vartheta.$$

It follows that  $p'^*s_{1!} = c_{1!}d_1^*$ ,  $p'^*s_{2!} = c_{2!}d_2^*$ . Now let  $L \in \mathcal{D}(Z_s)$ ,  $L' \in \mathcal{D}(\tilde{\mathcal{B}}^2)$ ,  $\tilde{L}' \in \mathcal{D}(\tilde{\mathcal{B}}^2)$ . We have  $\eta^*L \boxtimes L' \in \mathcal{D}(\mathcal{Y} \times \tilde{\mathcal{B}}^2)$ ,  $\tilde{L}' \boxtimes \eta^*L \in \mathcal{D}(\tilde{\mathcal{B}}^2 \times \mathcal{Y})$ . We have

$$p'_{12}{}^*\mathbf{b}'(L) \otimes p'_{23}{}^*L' = p'^*s_{1!}(\eta^*L \boxtimes L') = c_{1!}d_1^*(\eta^*L \boxtimes L') = c_{1!}(e_1^*L \boxtimes e_1'^*L'),$$

$$p'_{12}{}^*\tilde{L}' \otimes p'_{23}{}^*\mathbf{b}'(L) = p'^*s_{2!}(\tilde{L}' \boxtimes \eta^*L) = c_{2!}d_2^*(\tilde{L}' \boxtimes \eta^*L) = c_{2!}(e_2'^*\tilde{L}' \boxtimes e_1^*L),$$

where

$$e_1 : R \rightarrow Z_s \text{ is } (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) \mapsto \epsilon_s(x_1\mathbf{U}, x_2\mathbf{U}),$$

$$e_1' : R \rightarrow \tilde{\mathcal{B}}^2 \text{ is } (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) \mapsto (\gamma x_0\tau^{-s}\mathbf{U}, x_3\mathbf{U}),$$

$$e_2' : R \rightarrow \tilde{\mathcal{B}}^2 \text{ is } (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) \mapsto (x_0\mathbf{U}, \gamma^{-1}x_3\tau^s\mathbf{U}).$$

Applying  $p'_{13!}$  we see that

$$\mathbf{b}'(L) \circ L' = \tilde{c}_!(e_1^*L \boxtimes e_1'^*L), \tilde{L}' \circ \mathbf{b}'(L) = \tilde{c}_!(e_2'^*L \boxtimes e_1^*L),$$

where  $\tilde{c} : R \rightarrow \tilde{\mathcal{B}}^2$  is  $(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) \mapsto (x_0\mathbf{U}, x_3\mathbf{U})$ .

We define  $\mathbf{e} : \tilde{\mathcal{B}}^2 \rightarrow \tilde{\mathcal{B}}^2$  by  $\mathbf{e}(x\mathbf{U}, y\mathbf{U}) = (\mathbf{e}(x)\mathbf{U}, \mathbf{e}(y)\mathbf{U})$ . We show:

(i) *If in addition  $L' \in \mathcal{M}(\tilde{\mathcal{B}}^2)$  is  $G$ -equivariant, then we have canonically*

$$\mathbf{b}'(L) \circ L' = (\mathbf{e}^{s*}L') \circ \mathbf{b}'(L).$$

We take  $\tilde{L}' = \mathbf{e}^{s*}L'$ . It is enough to show that  $\tilde{c}_!(e_1^*L \boxtimes e_1'^*L') = \tilde{c}_!(e_2'^*\tilde{L}' \boxtimes e_1^*L)$ . Hence it is enough to show that we have canonically  $e_1'^*L' = e_2'^*\tilde{L}'$  that is,  $e_1'^*L' = e_2''^*L'$  where  $e_2'' = \mathbf{e}^s e_2' : R \rightarrow \tilde{\mathcal{B}}^2$ . We identify  $\tilde{G}_s$  with  $G$  by  $\gamma \mapsto g$  where  $\gamma = g\tau^s$ . Then  $e_1' : R \rightarrow \tilde{\mathcal{B}}^2$  is  $(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) \mapsto (g\mathbf{e}^s(x_0)\mathbf{U}, x_3\mathbf{U})$ ,  $e_2'' : R \rightarrow \tilde{\mathcal{B}}^2$  is  $(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) \mapsto (\mathbf{e}^s(x_0)\mathbf{U}, g^{-1}x_3\mathbf{U})$ . The equality  $e_1'^*L' = e_2''^*L'$  follows from the  $G$ -equivariance of  $L'$ . This proves (i).

We show:

(j) *If  $L \in \mathcal{C}_0^s Z_s$ ,  $L' \in \mathcal{C}^s \tilde{\mathcal{B}}^2$ , then we have canonically  $\underline{\mathbf{b}}(L) \circ L' = (\mathbf{e}^{s*}L') \circ \underline{\mathbf{b}}(L)$ .*

By (d), it is enough to prove that  $\underline{\mathbf{b}}'(L) \circ L' = (\mathbf{e}^{s*}L') \circ \underline{\mathbf{b}}'(L)$ . Using (i) together with (a),(b),(c) and results in [L16, 2.23], we see that both sides are equal to

$$\begin{aligned} & \underline{gr}_{\rho+\nu+3a}(\tilde{c}_!(e_1^*L \otimes e_1'^*L'))^{\rho+\nu+3a}((\rho+\nu+3a)/2) \\ & = \underline{gr}_{\rho+\nu+3a}\tilde{c}_!(e_1^*L \otimes e_2''^*L')^{\rho+\nu+3a}((\rho+\nu+3a)/2). \end{aligned}$$



**4.15.** Let

$$\mathfrak{Z}_s = \{(z_0\mathbf{U}, z_1\mathbf{U}, z_2\mathbf{U}, z_3\mathbf{U}), \gamma\} \in \tilde{\mathcal{B}}^4 \times \tilde{G}_s; \gamma \in z_2\mathbf{B}\tau^s z_1^{-1}\}.$$

Define  $\tilde{\vartheta} : \mathfrak{Z}_s \rightarrow \tilde{\mathcal{B}}^4$  by  $((z_0\mathbf{U}, z_1\mathbf{U}, z_2\mathbf{U}, z_3\mathbf{U}), \gamma) \mapsto (z_0\mathbf{U}, z_1\mathbf{U}, z_2\mathbf{U}, z_3\mathbf{U})$ . Let

$$' \mathcal{Y} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^5 \times \tilde{G}_s; \gamma \in x_3\mathbf{U}\tau^s x_0^{-1}, \gamma \in x_2\mathbf{B}\tau^s x_1^{-1}\},$$

$$'' \mathcal{Y} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^5 \times \tilde{G}_s; \gamma \in x_4\mathbf{U}\tau^s x_1^{-1}, \gamma \in x_3\mathbf{B}\tau^s x_2^{-1}\}.$$

Define  $'\vartheta : ' \mathcal{Y} \rightarrow \tilde{\mathcal{B}}^5$ ,  $''\vartheta : '' \mathcal{Y} \rightarrow \tilde{\mathcal{B}}^5$  by

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}).$$

We have isomorphisms  $'\mathfrak{c} : ' \mathcal{Y} \xrightarrow{\sim} \mathfrak{Z}_s$ ,  $''\mathfrak{c} : '' \mathcal{Y} \xrightarrow{\sim} \mathfrak{Z}_s$  given by

$$' \mathfrak{c} : ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_4\mathbf{U}), \gamma),$$

$$'' \mathfrak{c} : ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto ((x_0\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma).$$

Define  $'d : \tilde{\mathcal{B}}^5 \rightarrow \tilde{\mathcal{B}}^4$ ,  $''d : \tilde{\mathcal{B}}^5 \rightarrow \tilde{\mathcal{B}}^4$  by

$$'d : (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}) \mapsto (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_4\mathbf{U}),$$

$$''d : (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}) \mapsto (x_0\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}).$$

We fix  $w, u$  in  $W$  and  $\lambda, \lambda'$  in  $\mathfrak{s}_n$ . We assume that  $w \cdot \lambda \in I_n^s$ . The smooth subvarieties

$$' \mathcal{U} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \in ' \mathcal{Y}; x_1^{-1}x_2 \in G_w, x_3^{-1}x_4 \in G_{\mathbf{e}^s(u)}\},$$

$$\mathcal{U} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \in \mathfrak{Z}_s; x_1^{-1}x_2 \in G_w, x_0^{-1}g^{-1}x_3 \in G_u\},$$

$$'' \mathcal{U} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \in '' \mathcal{Y}; x_2^{-1}x_3 \in G_w, x_0^{-1}x_1 \in G_u\},$$

of  $' \mathcal{Y}, \mathfrak{Z}_s, '' \mathcal{Y}$  correspond to each other under the isomorphisms  $' \mathcal{Y} \xrightarrow{'\mathfrak{c}} \mathfrak{Z}_s \xleftarrow{''\mathfrak{c}} '' \mathcal{Y}$ . Moreover, the maps  $'\sigma : ' \mathcal{U} \rightarrow Z_s$ ,  $\sigma : \mathcal{U} \rightarrow Z_s$ ,  $''\sigma : '' \mathcal{U} \rightarrow Z_s$  given by

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto \epsilon_s(x_1\mathbf{U}, x_2\mathbf{U}),$$

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \mapsto \epsilon_s(x_1\mathbf{U}, x_2\mathbf{U}),$$

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto \epsilon_s(x_2\mathbf{U}, x_3\mathbf{U}),$$

correspond to each other under the isomorphisms  $' \mathcal{Y} \xrightarrow{'\mathfrak{c}} \mathfrak{Z}_s \xleftarrow{''\mathfrak{c}} '' \mathcal{Y}$ .

Also, the maps  $'\tilde{\sigma} : '\mathcal{U} \rightarrow \tilde{\mathcal{O}}_{\mathbf{e}^s(u)}$ ,  $\tilde{\sigma} : \mathcal{U} \rightarrow \tilde{\mathcal{O}}_{\mathbf{e}^s(u)}$ , given by

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto (x_3\mathbf{U}, x_4\mathbf{U}),$$

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \mapsto (\gamma x_0 \tau^{-s} \mathbf{U}, x_3\mathbf{U})$$

correspond to each other under the isomorphism  $'\mathcal{Y} \xrightarrow{'\epsilon} \mathfrak{Z}_s$  and the maps  $\tilde{\sigma}_1 : \mathcal{U} \rightarrow \tilde{\mathcal{O}}_u$ ,  $''\tilde{\sigma} : ''\mathcal{U} \rightarrow \tilde{\mathcal{O}}_u$  given by

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \mapsto (x_0\mathbf{U}, \gamma^{-1} x_3 \tau^s \mathbf{U}),$$

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto (x_0\mathbf{U}, x_1\mathbf{U}),$$

correspond to each other under the isomorphism  $\mathfrak{Z}_s \xleftarrow{''\epsilon} ''\mathcal{Y}$ . It follows that the local systems  $'\sigma^* \mathcal{L}_{\lambda,s}^{\dot{w}}$ ,  $\sigma^* \mathcal{L}_{\lambda,s}^{\dot{w}}$ ,  $''\sigma^* \mathcal{L}_{\lambda,s}^{\dot{w}}$  correspond to each other under the isomorphisms  $'\mathcal{Y} \xrightarrow{'\epsilon} \mathfrak{Z}_s \xleftarrow{''\epsilon} ''\mathcal{Y}$ ; the local systems  $'\tilde{\sigma}^* L_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}$ ,  $\tilde{\sigma}^* L_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}$  correspond to each other under the isomorphism  $'\mathcal{Y} \xrightarrow{'\epsilon} \mathfrak{Z}_s$ ; the local systems  $\tilde{\sigma}_1^* L_{\lambda'}^{\dot{u}}$ ,  $''\tilde{\sigma}^* L_{\lambda'}^{\dot{u}}$  correspond to each other under the isomorphism  $\mathfrak{Z}_s \xleftarrow{''\epsilon} ''\mathcal{Y}$ . Moreover, by the  $G$ -equivariance of  $L_{\lambda'}^{\dot{u}}$ , we have as in the proof of 4.14(i):  $\tilde{\sigma}^* L_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})} = \tilde{\sigma}_1^*(L_{\lambda'}^{\dot{u}})$ .

Let  $'K, K, ''K$  be the intersection cohomology complex of the closure of  $'\mathcal{U}, \mathcal{U}, ''\mathcal{U}$  respectively with coefficients in the local system

$$' \sigma^* \mathcal{L}_{\lambda,s}^{\dot{w}} \otimes ' \tilde{\sigma}^* L_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}, \sigma^* \mathcal{L}_{\lambda,s}^{\dot{w}} \otimes \tilde{\sigma}^* L_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})} = \sigma^* \mathcal{L}_{\lambda,s}^{\dot{w}} \otimes \tilde{\sigma}_1^*(L_{\lambda'}^{\dot{u}}), '' \sigma^* \mathcal{L}_{\lambda,s}^{\dot{w}} \otimes '' \tilde{\sigma}^* L_{\lambda'}^{\dot{u}},$$

on  $'\mathcal{U}, \mathcal{U}, ''\mathcal{U}$  (respectively), extended by 0 on the complement of this closure in  $'\mathcal{Y}, \mathfrak{Z}_s, ''\mathcal{Y}$ . We see that  $'K, K, ''K$  correspond to each other under the isomorphisms  $'\mathcal{Y} \xrightarrow{'\epsilon} \mathfrak{Z}_s \xleftarrow{''\epsilon} ''\mathcal{Y}$ . Hence we have  $'\mathbf{c}_1('K) = K = ''\mathbf{c}_1(''K)$ . Using this and the commutative diagram

$$\begin{array}{ccccc} '\mathcal{Y} & \xrightarrow{'\epsilon} & \mathfrak{Z}_s & \xleftarrow{''\epsilon} & ''\mathcal{Y} \\ '\vartheta \downarrow & & \tilde{\vartheta} \downarrow & & ''\vartheta \downarrow \\ \tilde{\mathcal{B}}^5 & \xrightarrow{'d} & \tilde{\mathcal{B}}^4 & \xleftarrow{''d} & \tilde{\mathcal{B}}^5 \end{array}$$

we see that

$$(a) \quad 'd_!'\vartheta_!('K) = ''d_!''\vartheta_!(''K).$$

(Both sides are equal to  $\tilde{\vartheta}_!K$ .)

**4.16.** In this subsection we study the functor  $'d_! : \mathcal{D}_m(\tilde{\mathcal{B}}^5) \rightarrow \mathcal{D}_m(\tilde{\mathcal{B}}^4)$ . Let  $\mathbf{w} = (w_1, w_2, w_3, w_4)$ ,  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathfrak{s}_n^4$ ,  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$  (with  $\omega_i \in \kappa_0^{-1}(w_i)$ ). Assume that  $w_4 \cdot \lambda_4 \preceq \mathbf{c}$ . Let  $K = M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega}, [1,4]} \langle |\mathbf{w}| + 5\rho + \nu \rangle \in \mathcal{D}_m(\tilde{\mathcal{B}}^5)$ . As in [L16, 3.16], properties (a)(b),(c),(d) hold:

(a) If  $h > a + \rho$  then  $('d_!K)^h \in {}'\mathcal{M}^{\prec}(\tilde{\mathcal{B}}^4)$ . Moreover,

$$\begin{aligned} \underline{gr_{a+\rho}(({}'d_!K)^{a+\rho})((a+\rho)/2)} &= \oplus_{y' \in W; y'^{-1} \cdot \lambda_4 \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2}(\mathbf{L}_{\lambda_4}^{\dot{y}'^{-1}}, \mathbf{L}_{\lambda_3}^{\omega_3} \circ \mathbf{L}_{\lambda_4}^{\omega_4}) \\ &\otimes M_{\lambda_1, \lambda_2, \lambda_4}^{\omega_1, \omega_2, \dot{y}'^{-1}, [1,3]} \langle |w_1| + |w_2| + |y'| + 4\rho + \nu \rangle. \end{aligned}$$

(b) If  $K \in {}_4\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^5)$  then  $'d_!(K) \in {}_4\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$ .

(c) If  $K \in {}_4\mathcal{D}^{\prec}(\tilde{\mathcal{B}}^5)$  then  $'d_!(K) \in {}_4\mathcal{D}^{\prec}(\tilde{\mathcal{B}}^4)$ .

(d) If  $K \in {}_4\mathcal{M}^{\preceq}(\tilde{\mathcal{B}}^5)$  and  $h > a + \rho$  then  $('d_!(K))^h \in {}_4\mathcal{M}^{\prec}(\tilde{\mathcal{B}}^4)$ .

**4.17.** In this subsection we study the functor  $''d_! : \mathcal{D}_m(\tilde{\mathcal{B}}^5) \rightarrow \mathcal{D}_m(\tilde{\mathcal{B}}^4)$ . Let  $\mathbf{w} = (w_1, w_2, w_3, w_4)$ ,  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathfrak{s}_n^4$ ,  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$  (with  $\omega_i \in \kappa_0^{-1}(w_i)$ ). Assume that  $w_1 \cdot \lambda_1 \preceq \mathbf{c}$ . Let  $K = M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega}, [1,4]} \langle |\mathbf{w}| + 5\rho + \nu \rangle \in \mathcal{D}_m(\tilde{\mathcal{B}}^5)$ . As in [L16, 3.17], properties (a)(b),(c),(d) hold:

(a) If  $h > a + \rho$  then  $''d_!K)^h \in {}'\mathcal{M}^{\prec}(\tilde{\mathcal{B}}^4)$ . Moreover,

$$\begin{aligned} \underline{gr_{a+\rho}(({}''d_!K)^{a+\rho})((a+\rho)/2)} &= \oplus_{y' \in W; y' \cdot \lambda_2 \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2}(\mathbf{L}_{\lambda_2}^{\dot{y}'}, \mathbf{L}_{\lambda_1}^{\omega_1} \circ \mathbf{L}_{\lambda_2}^{\omega_2}) \\ &\otimes M_{\lambda_2, \lambda_3, \lambda_4}^{\dot{y}', \omega_3, \omega_4, [1,3]} \langle |w_3| + |w_4| + |y'| + 4\rho + \nu \rangle. \end{aligned}$$

(b) If  $K \in {}_1\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^5)$  then  $''d_!(K) \in {}_1\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$ .

(c) If  $K \in {}_1\mathcal{D}^{\prec}(\tilde{\mathcal{B}}^5)$  then  $''d_!(K) \in {}_1\mathcal{D}^{\prec}(\tilde{\mathcal{B}}^4)$ .

(d) If  $K \in {}_1\mathcal{M}^{\preceq}(\tilde{\mathcal{B}}^5)$  and  $h > a + \rho$  then  $''d_!(K))^h \in {}_1\mathcal{M}^{\prec}(\tilde{\mathcal{B}}^4)$ .

**4.18.** Let  $w \cdot \lambda \in I_n^s$ ,  $u \cdot \lambda' \in \mathbf{c}$ . We shall apply the method of [L14, 1.12] with  $\Phi : \mathcal{D}_m(Y_1) \rightarrow \mathcal{D}_m(Y_2)$  replaced by  $'d_! : \mathcal{D}_m(\tilde{\mathcal{B}}^5) \rightarrow \mathcal{D}_m(\tilde{\mathcal{B}}^4)$  and with  $\mathcal{D}^{\preceq}(Y_1)$ ,  $\mathcal{D}^{\preceq}(Y_2)$  replaced by  ${}_4\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^5)$ ,  ${}_4\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$ , see 4.15. We shall take  $\mathbf{X}$  in *loc.cit.* equal to  $\Xi = {}'\vartheta_!({}'K)$  as in 4.15,  $(w_2, w_4) = (w, \mathbf{e}^s(u))$ ,  $(\lambda_2, \lambda_4) = (\lambda, \mathbf{e}^s(\lambda'))$ . The conditions of *loc.cit.* are satisfied: those concerning  $\mathbf{X}$  are satisfied with  $c' = k = |w| + |u| + 3\nu + 5\rho$  (see 4.8(c)); those concerning  $\Phi$  are satisfied with  $c = a + \rho$  (see 4.16). We see that

$$\begin{aligned} &\underline{gr_{a+\rho+k}(({}'d_!{}'\vartheta_!({}'K))^{a+\rho+k})((a+\rho+k)/2)} \\ &= \underline{gr_{a+\rho}(({}'d_!gr_k(({}'\vartheta_!({}'K))^k)(k/2))^{a+\rho})((a+\rho)/2)}. \end{aligned}$$

Using 4.11(a), we have:

$$\begin{aligned} gr_k({}'\vartheta_!({}'K))^k(k/2) &= \oplus_{y \in W} M_{\mathbf{e}^{-s}(\lambda), \lambda, y(\lambda), \mathbf{e}^s(\lambda')}^{\mathbf{e}^{-s}(\dot{y}), \dot{w}, \dot{y}^{-1}, \mathbf{e}^s(\dot{u}), [1,4]} \langle 2|y| + |w| + |u| + 5\rho + \nu \rangle \\ &= \underline{gr_k({}'\vartheta_!({}'K))^k(k/2)}. \end{aligned}$$

Hence, using 4.16(a), we have

$$\begin{aligned} & \underline{gr_{a+\rho}(({}^l d_! gr_k(({}^l \vartheta_!({}^l K))^k)(k/2))^{a+\rho})((a+\rho)/2)} \\ &= \bigoplus_{y \in W} \bigoplus_{y' \in W; y'^{-1} \cdot \mathbf{e}^s(\lambda') \in \mathbf{c}} \text{Hom}_{\mathcal{C}^c \tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^s(\lambda')}^{\dot{y}'^{-1}}, \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \circ \mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}) \\ & \otimes M_{\mathbf{e}^{-s}(\lambda), \lambda, \mathbf{e}^s(\lambda')}^{\mathbf{e}^{-s}(\dot{y}), \dot{w}, \dot{y}'^{-1}, [1,3]} \langle |y| + |w| + |y'| + 4\rho + \nu \rangle. \end{aligned}$$

Since  $y'^{-1} \cdot \mathbf{e}^s(\lambda') \in \mathbf{c}$ ,  $\mathbf{e}^s(u) \cdot \mathbf{e}^s(\lambda') \in \mathbf{c}$  (recall that  $\mathbf{e}^s \mathbf{c} = \mathbf{c}$ ), for  $y \in W$  we have

$$\text{Hom}_{\mathcal{C}^c \tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^s(\lambda')}^{\dot{y}'^{-1}}, \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \circ \mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}) = 0$$

unless  $\mathbf{e}^s(\lambda') = y'(\lambda)$  (see [L16, 4.6(b)]) and  $y^{-1} \cdot y(\lambda) \in \mathbf{c}$  (see [L16, 2.26(a)]) or equivalently,  $y \cdot \lambda \in \mathbf{c}$ . Thus we have

$$\begin{aligned} & \underline{gr_{a+\rho+k}(({}^l d_! {}^l \vartheta_!({}^l K))^{a+\rho+k})((a+\rho+k)/2)} \\ &= \bigoplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \bigoplus_{y' \in W; y'^{-1} \cdot y'(\lambda) \in \mathbf{c}} \text{Hom}_{\mathcal{C}^c \tilde{\mathcal{B}}^2}(\mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}, \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \circ \mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}) \\ & \otimes M_{\mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{w}, \dot{y}'^{-1}, [1,3]} \langle |y| + |w| + |y'| + 4\rho + \nu \rangle. \text{taga} \end{aligned}$$

**4.19.** In the setup of 4.18 we shall apply the method of [L14, 1.12] with  $\Phi : \mathcal{D}_m(Y_1) \rightarrow \mathcal{D}_m(Y_2)$  replaced by  ${}^l d_! : \mathcal{D}_m(\tilde{\mathcal{B}}^5) \rightarrow \mathcal{D}_m(\tilde{\mathcal{B}}^4)$  and with  $\mathcal{D}^\preceq(Y_1)$ ,  $\mathcal{D}^\preceq(Y_2)$  replaced by  ${}_1 \mathcal{D}^\preceq(\tilde{\mathcal{B}}^5)$ ,  ${}_1 \mathcal{D}^\preceq(\tilde{\mathcal{B}}^4)$ , see 4.15. We shall take  $\mathbf{X}$  in *loc.cit.* equal to  $\Xi = {}^l \vartheta_!({}^l K)$  as in 4.15,  $(w_1, w_3) = (u, w)$ ,  $(\lambda_1, \lambda_3) = (\lambda', \lambda)$ . The conditions of *loc.cit.* are satisfied: those concerning  $\mathbf{X}$  are satisfied with  $c' = k = |w| + |u| + 3\nu + 5\rho$  (see 4.8(c)); those concerning  $\Phi$  are satisfied with  $c = a + \rho$  (see 4.17). We see that

$$\begin{aligned} & \underline{gr_{a+\rho+k}(({}^l d_! {}^l \vartheta_!({}^l K))^{a+\rho+k})((a+\rho+k)/2)} \\ &= \underline{gr_{a+\rho}(({}^l d_! gr_k(({}^l \vartheta_!({}^l K))^k)(k/2))^{a+\rho})((a+\rho)/2)}. \end{aligned}$$

Using 4.11(a), we have:

$$\begin{aligned} & gr_k({}^l \vartheta_!({}^l K))^k(k/2) = \bigoplus_{y' \in W} M_{\lambda', \mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)}^{\dot{u}, \mathbf{e}^{-s}(\dot{y}'), \dot{w}, \dot{y}'^{-1}, [1,4]} \langle 2|y'| + |w| + |u| + 5\rho + \nu \rangle \\ &= \underline{gr_k({}^l \vartheta_!({}^l K))^k(k/2)}. \end{aligned}$$

Hence, using 4.17(a), we have

$$\begin{aligned} & \underline{gr_{a+\rho}(({}^l d_! gr_k(({}^l \vartheta_!({}^l K))^k)(k/2))^{a+\rho})((a+\rho)/2)} \\ &= \bigoplus_{y' \in W} \bigoplus_{y_1 \in W; y_1 \cdot \mathbf{e}^{-s}(\lambda) \in \mathbf{c}} \text{Hom}_{\mathcal{C}^c \tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\dot{y}_1}, \mathbf{L}_{\lambda'}^{\dot{u}} \circ \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')} ) \\ & \otimes M_{\mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)}^{\dot{y}_1, \dot{w}, \dot{y}'^{-1}, [1,3]} \langle |y_1| + |w| + |y'| + 4\rho + \nu \rangle. \end{aligned}$$

Since  $u \cdot \lambda' \in \mathbf{c}$ , for  $y' \in W$  we have

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{(\mathbf{e}^{-s}(\lambda))}^{\dot{y}_1}, \mathbf{L}_{\lambda' \circ \mathbf{e}^{-s}(\lambda)}^{\dot{y}_1} \circ \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')} ) = 0$$

unless  $\mathbf{e}^s(\lambda') = y'(\lambda)$  (see [L16, 4.6(b)]) and  $y'(\lambda) = \mathbf{e}^s(\lambda')$  (see [L16, 2.26(a)]). Thus we have

$$\begin{aligned} & \underline{gr_{a+\rho+k}(({}''d_!''\vartheta_!({}''K))^{a+\rho+k})((a+\rho+k)/2)} \\ &= \bigoplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \bigoplus_{y_1 \in W; y_1 \cdot \mathbf{e}^{-s}(\lambda) \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\dot{y}_1}, \mathbf{L}_{\lambda' \circ \mathbf{e}^{-s}(\lambda)}^{\dot{y}_1} \circ \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')} ) \\ & \otimes M_{\mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)}^{\dot{y}_1, \dot{w}, \dot{y}'^{-1}, [1,3]} \langle |y_1| + |w| + |y'| + 4\rho + \nu \rangle. \end{aligned}$$

Setting  $y_1 = \mathbf{e}^{-s}y$  and using that  $\mathbf{e}^{-s}y \cdot \mathbf{e}^{-s}(\lambda) \in \mathbf{c}$  if and only if  $y \cdot \lambda \in \mathbf{c}$ , we can rewrite this as follows:

$$\begin{aligned} & \underline{gr_{a+\rho+k}(({}''d_!''\vartheta_!({}''K))^{a+\rho+k})((a+\rho+k)/2)} \\ &= \bigoplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \bigoplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}\dot{y}}, \mathbf{L}_{\lambda' \circ \mathbf{e}^{-s}(\lambda)}^{\dot{y}} \circ \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')} ) \\ (a) \quad & \otimes M_{\mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)}^{\mathbf{e}^{-s}\dot{y}, \dot{w}, \dot{y}'^{-1}, [1,3]} \langle |y| + |w| + |y'| + 4\rho + \nu \rangle. \end{aligned}$$

**4.20.** Let  $y_1 \cdot \lambda_1 \in \mathbf{c}$ ,  $y_2 \cdot \lambda_2 \in \mathbf{c}$ ,  $y_3 \cdot \lambda_3 \in \mathbf{c}$ . From [L16, 3.20] we see that:

(a) *we have canonically*

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{y_2(\lambda_2)}^{\dot{y}_2^{-1}}, \mathbf{L}_{y_1(\lambda_1)}^{\dot{y}_1^{-1}} \circ \mathbf{L}_{\lambda_3}^{\dot{y}_3}) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{\lambda_1}^{\dot{y}_1}, \mathbf{L}_{\lambda_3 \circ \lambda_2}^{\dot{y}_3} \circ \mathbf{L}_{\lambda_2}^{\dot{y}_2}).$$

In the setup of 4.18, we apply 4.18(a), 4.19(a) to  $w \cdot \lambda$ ,  $u \cdot \lambda'$  and we use the equality

$$\begin{aligned} & \underline{gr_{a+\rho+k}({}'d_!'\vartheta_!({}'K))^{a+\rho+k})((a+\rho+k)/2)} \\ &= \underline{gr_{a+\rho+k}({}''d_!''\vartheta_!({}''K))^{a+\rho+k})((a+\rho+k)/2)} \end{aligned}$$

which comes from  $'d_!'\vartheta_!({}'K) = {}''d_!''\vartheta_!({}''K)$ , see 4.15(a); we obtain

$$\begin{aligned} & \bigoplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \bigoplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}, \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \circ \mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}) \\ & \otimes M_{\mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)}^{\mathbf{e}^{-s}(\dot{y}), \dot{w}, \dot{y}'^{-1}, [1,3]} \langle |y| + |w| + |y'| + 4\rho + \nu \rangle \\ &= \bigoplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \bigoplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}\dot{y}}, \mathbf{L}_{\lambda' \circ \mathbf{e}^{-s}(\lambda)}^{\dot{y}} \circ \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')} ) \\ (b) \quad & \otimes M_{\mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)}^{\mathbf{e}^{-s}\dot{y}, \dot{w}, \dot{y}'^{-1}, [1,3]} \langle |y| + |w| + |y'| + 4\rho + \nu \rangle. \end{aligned}$$

**4.21.** We assume that  $w \cdot \lambda, u \cdot \lambda'$  in 4.18 satisfy in addition  $w \cdot \lambda \in \mathbf{c}$ . We apply  $p_{03}!$  and  $\langle N \rangle$  for some  $N$  to the two sides of 4.20(b). (Recall that  $p_{03} : \tilde{\mathcal{B}}^4 \rightarrow \tilde{\mathcal{B}}^2$ .) We obtain

$$\begin{aligned} & \oplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \oplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2}(\mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}, \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \circ \mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}) \\ & \otimes \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \circ \mathbf{L}_{\lambda}^{\dot{w}} \circ \mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}} \\ & = \oplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \oplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}\dot{y}}, \mathbf{L}_{\lambda'}^{\dot{u}} \circ \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')} ) \\ & \otimes \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}\dot{y}} \circ \mathbf{L}_{\lambda}^{\dot{w}} \circ \mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}. \end{aligned}$$

Applying  $\underline{()^{\{2(a-\nu)\}}}$  to both sides and using [L16, 2.24(a)] we obtain

$$\begin{aligned} & \oplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \oplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2}(\mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}, \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \circ \mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}) \\ & \otimes \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \circ \mathbf{L}_{\lambda}^{\dot{w}} \circ \mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}} \\ & = \oplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \oplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}\dot{y}}, \mathbf{L}_{\lambda'}^{\dot{u}} \circ \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')} ) \\ & \otimes \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}\dot{y}} \circ \mathbf{L}_{\lambda}^{\dot{w}} \circ \mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}, \end{aligned}$$

or equivalently

$$\begin{aligned} & \oplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \circ \mathbf{L}_{\lambda}^{\dot{w}} \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \circ \mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})} \\ & = \oplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \mathbf{L}_{\lambda'}^{\dot{u}} \circ \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')} \circ \mathbf{L}_{\lambda}^{\dot{w}} \circ \mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}. \end{aligned}$$

Using 4.13(d), this can be rewritten as follows:

$$(a.) \quad \underline{\mathbf{h}}(\mathbb{L}_{\lambda,s}^{\dot{w}}) \circ \mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})} = \mathbf{L}_{\lambda'}^{\dot{u}} \circ \underline{\mathbf{h}}(\mathbb{L}_{\lambda,s}^{\dot{w}}).$$

Another identification of the two sides in (a) is given by 4.14(j) with  $L = \mathbb{L}_{\lambda,s}^{\dot{w}}$ ,  $L' = \mathbf{L}_{\lambda'}^{\dot{u}}$  (note that  $\underline{\mathbf{h}}(L) = \underline{\mathbf{h}}'(L)$  by 4.14(d)). In fact, the arguments in 4.13-4.20 and in this subsection show that

(b) *these two identifications of the two sides of (a) coincide.*

**4.22.** Let  $s', s'' \in \mathbf{Z}$ . Let

$$\begin{aligned} V &= \{(B_0, B_1, B_2, \gamma U_{B_0}, \gamma' U_{B_1}); \\ & (B_0, B_1, B_2) \in \mathcal{B}^3, \gamma \in \tilde{G}_{s'}, \gamma' \in \tilde{G}_{s''}, \gamma B_0 \gamma^{-1} = B_1, \gamma' B_1 \gamma'^{-1} = B_2\}. \end{aligned}$$

Define  $p_{01} : V \rightarrow Z_{s'}$ ,  $p_{12} : V \rightarrow Z_{s''}$ ,  $p_{02} : V \rightarrow Z_{s'+s''}$  by

$$p_{01} : (B_0, B_1, B_2, \gamma U_{B_0}, \gamma' U_{B_1}) \mapsto (B_0, B_1, \gamma U_{B_0}),$$

$$p_{12} : (B_0, B_1, B_2, gU_{B_0}, \gamma'U_{B_1}) \mapsto (B_1, B_2, \gamma'U_{B_1}),$$

$$p_{02} : (B_0, B_1, B_2, \gamma U_{B_0}, \gamma'U_{B_1}) \mapsto (B_0, B_2, \gamma'\gamma U_{B_0}).$$

For  $L \in \mathcal{D}(Z_{s'})$ ,  $L' \in \mathcal{D}(Z_{s''})$  we set

$$L \bullet L' = p_{02}!(p_{01}^*L \otimes p_{12}^*L') \in \mathcal{D}(Z_{s'+s''}).$$

This operation defines a monoidal structure on  $\sqcup_{s' \in \mathbf{Z}} \mathcal{D}(Z_{s'})$ . Hence if  ${}^1L \in \mathcal{D}(Z_{s_1})$ ,  ${}^2L \in \mathcal{D}(Z_{s_2})$ ,  $\dots$ ,  ${}^rL \in \mathcal{D}(Z_{s_r})$ , then  ${}^1L \bullet {}^2L \bullet \dots \bullet {}^rL \in \mathcal{D}(Z_{s_1+\dots+s_r})$  is well defined. Note that, if  $L \in \mathcal{D}_m(Z_{s'})$ ,  $L'_m \in \mathcal{D}(Z_{s''})$  then we have naturally  $L \bullet L' \in \mathcal{D}_m(Z_{s'+s''})$ . We show:

(a) For  $L \in \mathcal{D}(Z_{s'})$ ,  $L' \in \mathcal{D}(Z_{s''})$  we have canonically  $\epsilon_{s'+s''}^*(L \bullet L') = \epsilon_{s'}^*(L) \circ \epsilon_{s''}^*(L')$ .

Let

$$Y = \{(x\mathbf{U}, y\mathbf{U}, \gamma U_{x\mathbf{B}x^{-1}}); x\mathbf{U} \in \tilde{\mathcal{B}}, y\mathbf{U} \in \tilde{\mathcal{B}}; \gamma \in \tilde{G}_{s'}\}.$$

Define  $j : Y \rightarrow \tilde{\mathcal{B}}^2$ ,  $j_1 : Y \rightarrow Z_{s'}$ ,  $j_2 : Y \rightarrow Z_{s''}$  by

$$\begin{aligned} j(x\mathbf{U}, y\mathbf{U}, \gamma U_{x\mathbf{B}x^{-1}}) &= (x\mathbf{U}, y\mathbf{U}), \\ j_1(x\mathbf{U}, y\mathbf{U}, \gamma U_{x\mathbf{B}x^{-1}}) &= (x\mathbf{B}x^{-1}, \gamma x\mathbf{B}x^{-1}\gamma^{-1}, \gamma U_{x\mathbf{B}x^{-1}}), \\ j_2(x\mathbf{U}, y\mathbf{U}, \gamma U_{x\mathbf{B}x^{-1}}) &= (\gamma x\mathbf{B}x^{-1}\gamma^{-1}, y\mathbf{B}y^{-1}, y\mathbf{U}\tau^{s'+s''}x^{-1}\gamma^{-1}). \end{aligned}$$

From the definitions we have

$$\epsilon_{s'+s''}^*(L \bullet L') = j!(j_1^*(L) \otimes j_2^*(L')) = \epsilon_{s'}^*(L) \circ \epsilon_{s''}^*(L')$$

and (a) follows.

**4.23.** Let  $s' \in \mathbf{Z}_{\mathbf{c}}$ . Let  $L \in \mathcal{D}^\spadesuit Z_s$ ,  $L' \in \mathcal{D}^\spadesuit Z_{s'}$ . We show:

(a) If  $L \in \mathcal{D}^\preceq Z_s$  or  $L' \in \mathcal{D}^\preceq Z_{s'}$  then  $L \bullet L' \in \mathcal{D}^\preceq Z_{s+s'}$ . If  $L \in \mathcal{D}^\prec Z_s$  or  $L' \in \mathcal{D}^\prec Z_{s'}$  then  $L \bullet L' \in \mathcal{D}^\prec Z_{s+s'}$ .

For the first assertion of (a) we can assume that  $L = \mathbb{L}_{\lambda, s}^{\dot{w}}$ ,  $L' = \mathbb{L}_{\lambda', s'}^{\dot{w}'}$  with  $w \cdot \lambda \in I_n^s$ ,  $w' \cdot \lambda' \in I_n^{s'}$  and either  $w \cdot \lambda \preceq \mathbf{c}$  or  $w' \cdot \lambda' \preceq \mathbf{c}$ . Assume that  $w_1 \cdot \lambda_1 \in I_n^{s+s'}$  and  $\mathbb{L}_{\lambda_1, s+s'}^{\dot{w}_1}$  is a composition factor of  $(L \bullet L')^j$ . Then  $\mathbf{L}_{\lambda_1}^{\dot{w}_1} = \tilde{\epsilon}_{s+s'} \mathbb{L}_{\lambda_1, s+s'}^{\dot{w}_1}$  is a composition factor of

$$\begin{aligned} \epsilon_{s+s'}^*(L \bullet L')^j \langle \rho \rangle &= (\epsilon_{s+s'}^*(L \bullet L'))^{j+\rho}(\rho/2) = (\epsilon_s^*L \circ \epsilon_{s'}^*L')^{j+\rho}(\rho/2) \\ &= (\epsilon_s^*L \langle \rho \rangle \circ \epsilon_{s'}^*L' \langle \rho \rangle)^{j-\rho}(-\rho/2) = (\mathbf{L}_{\lambda}^{\dot{w}} \circ \mathbf{L}_{\lambda'}^{\dot{w}'})^{j-\rho}(\rho/2). \end{aligned}$$

From [L16, 2.23(b)] we see that  $w_1 \cdot \lambda_1 \preceq \mathbf{c}$ . This proves the first assertion of (a). The second assertion of (a) can be reduced to the first assertion.

We show:

(b) Assume that  $L \in \mathcal{M}^\bullet Z_s, L' \in \mathcal{M}^\bullet Z_{s'}$  and that either  $L \in \mathcal{D}^\preceq Z_s$  or  $L' \in \mathcal{D}^\preceq Z_{s'}$ . If  $j > a + \rho - \nu$  then  $(L \bullet L')^j \in \mathcal{M}^\prec Z_{s+s'}$ .

We can assume that  $L = \mathbb{L}_{\lambda,s}^{\dot{w}}, L' = \mathbb{L}_{\lambda',s'}^{\dot{w}'}$  with  $w \cdot \lambda \in I_n^s, w' \cdot \lambda' \in I_n^{s'}$  and either  $w \cdot \lambda \in \mathbf{c}$  or  $w' \cdot \lambda' \in \mathbf{c}$ . Assume that  $w_1 \cdot \lambda_1 \in I_n^{s+s'}$  and that  $\mathbb{L}_{\lambda_1,s+s'}^{\dot{w}_1}$  is a composition factor of  $(L \bullet L')^j$ . Then as in the proof of (a),  $\mathbf{L}_{\lambda_1}^{\dot{w}_1}$  is a composition factor of

$$\tilde{e}_{s+s'}(L \bullet L')^j = (\mathbf{L}_{\lambda}^{\dot{w}} \circ \mathbf{L}_{\lambda'}^{\dot{w}'})^{j-\rho}(-\rho/2).$$

Since  $j - \rho > a - \nu$  we see from [L16, 2.23(a)] that  $w_1 \cdot \lambda_1 \prec \mathbf{c}$ . This proves (b).

**4.24.** Let  $s' \in \mathbf{Z}_{\mathbf{c}}$ . For  $L \in \mathcal{C}_0^{\mathbf{c}} Z_s, L' \in \mathcal{C}_0^{\mathbf{c}} Z_{s'}$  we set

$$L \bullet L' = \underline{(L \bullet L')}^{\{a+\rho-\nu\}} \in \mathcal{C}_0^{\mathbf{c}} Z_{s+s'}.$$

Using 4.23(a),(b) we see as in [L16, 2.24] that for  $L \in \mathcal{C}_0^{\mathbf{c}} Z_s, L' \in \mathcal{C}_0^{\mathbf{c}} Z_{s'}, L'' \in \mathcal{C}_0^{\mathbf{c}} Z_{s''}$  we have

$$L \bullet (L' \bullet L'') = (L \bullet L') \bullet L'' = \underline{(L \bullet L' \bullet L'')}^{\{2a+2\rho-2\nu\}}.$$

We see that  $L, L' \mapsto L \bullet L'$  defines a monoidal structure on  $\sqcup_{s' \in \mathbf{Z}_{\mathbf{c}}} \mathcal{C}_0^{\mathbf{c}} Z_{s'}$ . Hence if  ${}^1 L \in \mathcal{C}_0^{\mathbf{c}} Z_{s_1}, {}^2 L \in \mathcal{C}_0^{\mathbf{c}} Z_{s_2}, \dots, {}^r L \in \mathcal{C}_0^{\mathbf{c}} Z_{s_r}$ , then  ${}^1 L \bullet {}^2 L \bullet \dots \bullet {}^r L \in \mathcal{C}_0^{\mathbf{c}} Z_{s_1+\dots+s_r}$  is well defined; we have

$$(a) \quad {}^1 L \bullet {}^2 L \bullet \dots \bullet {}^r L = \underline{({}^1 L \bullet {}^2 L \bullet \dots \bullet {}^r L)}^{\{(r-1)(a+\rho-\nu)\}}.$$

For  $L \in \mathcal{C}_0^{\mathbf{c}} Z_s, L' \in \mathcal{C}_0^{\mathbf{c}} Z_{s'}$  we have  $\tilde{e}_s L, \tilde{e}_{s'} L' \in \mathcal{C}_0^{\mathbf{c}} \tilde{\mathcal{B}}^2$ . We show:

$$(b) \quad \tilde{e}_{s+s'}(L \bullet L') = (\tilde{e}_s L) \bullet (\tilde{e}_{s'} L').$$

It is enough to show that

$$\begin{aligned} & \epsilon_{s+s'}^*(gr_0((L \bullet L')^{a+\rho-\nu})((a+\rho-\nu)/2))[\rho](\rho/2) \\ &= gr_0((\epsilon_s^* L[\rho](\rho/2) \circ \epsilon_{s'}^* L'[\rho](\rho/2))^{a-\nu}((a-\nu)/2))). \end{aligned}$$

The left hand side is equal to

$$gr_0(\epsilon_{s+s'}^*((L \bullet L')^{a+\rho-\nu})((a+\rho-\nu)/2))[\rho](\rho/2))$$

hence it is enough to show:

$$\begin{aligned} & \epsilon_{s+s'}^*((L \bullet L')^{a+\rho-\nu})((a+\rho-\nu)/2))[\rho](\rho/2) \\ &= (\epsilon_s^* L[\rho](\rho/2) \circ \epsilon_{s'}^* L'[\rho](\rho/2))^{a-\nu}((a-\nu)/2)) \end{aligned}$$

that is,

$$\epsilon_{s+s'}^*((L \bullet L')^{a+\rho-\nu})[\rho] = (\epsilon_s^* L[\rho] \circ \epsilon_{s'}^* L'[\rho])^{a-\nu},$$

or, after using 4.3(b):

$$(\epsilon_{s+s'}^*(L \bullet L'))^{a+2\rho-\nu} = (\epsilon_s^* L \circ \epsilon_{s'}^* L')^{a+2\rho-\nu}.$$

It remains to use that  $\epsilon_{s+s'}^*(L \bullet L') = \epsilon_s^* L \circ \epsilon_{s'}^* L'$ , see 4.22(a).



**4.25.** In the setup of 4.14 let

$${}^\diamond\mathcal{Y} = \mathbf{T}^2 \setminus \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^4 \times \tilde{G}_s; \gamma \in x_3\mathbf{U}\tau^s x_0^{-1}, \gamma \in x_2\mathbf{U}\tau^s x_1^{-1}\}$$

where  $\mathbf{T}^2$  acts freely by

$$(t_1, t_2) : ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \mapsto ((x_0t_1\mathbf{U}, x_1t_2\mathbf{U}, x_2t_2\mathbf{U}, x_3t_1\mathbf{U}), \gamma).$$

We define  ${}^\diamond\eta : {}^\diamond\mathcal{Y} \rightarrow Z_s$  by

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \mapsto \epsilon_s(x_1\mathbf{U}, x_2\mathbf{U}).$$

We define  $d : {}^\diamond\mathcal{Y} \rightarrow Z_s$  by

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \mapsto \epsilon_s(x_0\mathbf{U}, x_3\mathbf{U}).$$

We define  $\mathfrak{b}'' : \mathcal{D}(Z_s) \rightarrow \mathcal{D}(Z_s)$  and  $\mathfrak{b}'' : \mathcal{D}_m(Z_s) \rightarrow \mathcal{D}_m(Z_s)$  by

$$\mathfrak{b}''(L) = d_!({}^\diamond\eta)^* L.$$

From the definitions it is clear that

$$(a) \quad \mathfrak{b}'(L) = \epsilon_s^* \mathfrak{b}''(L).$$

Using (a) we see that 4.14(a),(b),(c) imply the following statements.

(b) If  $L \in \mathcal{D}^{\preceq}(Z_s)$ , then  $\mathfrak{b}''(L) \in \mathcal{D}^{\preceq}Z_s$ . If  $L \in \mathcal{D}^{\prec}(Z_s)$  then  $\mathfrak{b}''(L) \in \mathcal{D}^{\prec}Z_s$ .

(c) If  $L \in \mathcal{M}^{\preceq}(Z_s)$  and  $h > 2\nu + 2a$  then  $(\mathfrak{b}''(L))^h \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$ .

We define  $\underline{\mathfrak{b}}'' : \mathcal{C}_0^{\mathfrak{c}}(Z_s) \rightarrow \mathcal{C}_0^{\mathfrak{c}}(Z_s)$  by

$$\underline{\mathfrak{b}}''(L) = \underline{gr_{2\nu+2a}((\mathfrak{b}''(L))^{2\nu+2a})}(\nu + a).$$

Using results in 4.3 we see that, if  $L \in \mathcal{C}_0^{\mathfrak{c}}Z_s$ , then

$$(d) \quad \underline{\mathfrak{b}}'(L) = \tilde{\epsilon}_s(\underline{\mathfrak{b}}''(L)).$$

## 5. THE MONOIDAL CATEGORY $\mathcal{C}^{\mathfrak{c}}\tilde{\mathcal{B}}^2$

**5.1.** In this section,  $\mathbf{c}, a, \mathfrak{o}, n, \Psi$  are as in 3.1(a).

Define  $\delta : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}^2$  by  $x\mathbf{U} \mapsto (x\mathbf{U}, x\mathbf{U})$ . For  $w \cdot \lambda \in \mathbf{c}$  we set

$$\beta_{w \cdot \lambda} = \mathcal{H}^{-a+|w|}(\delta^*(L_{\lambda}^{\dot{w}\sharp}))((-a + |w|)/2).$$

By [L16, 4.1] we have

(a)  $\dim \beta_{w \cdot \lambda} = 1$  if  $w \cdot \lambda \in \mathbf{D}_{\mathbf{c}}$ ,  $\dim \beta_{w \cdot \lambda} = 0$  if  $w \cdot \lambda \notin \mathbf{D}_{\mathbf{c}}$ .

We set

$$\mathbf{1}' = \oplus_{d \cdot \lambda \in \mathbf{D}_{\mathbf{c}}} \beta_{d \cdot \lambda}^* \otimes \mathbf{L}_{\lambda}^d \in \mathcal{C}_0^{\mathfrak{c}}\tilde{\mathcal{B}}^2.$$

Here  $\beta_{d \cdot \lambda}^*$  is the vector space dual to  $\beta_{d \cdot \lambda}$ .

**5.2.** For  $L \in \mathcal{D}_m(\tilde{\mathcal{B}}^2)$  we set  $L^\dagger = \tilde{\mathfrak{h}}^* L$  where  $\tilde{\mathfrak{h}} : \tilde{\mathcal{B}}^2 \rightarrow \tilde{\mathcal{B}}^2$  is as in 3.1. By [L16, 4.4(b)], we have:

(a) *If  $L \in \mathcal{C}_0^c \tilde{\mathcal{B}}^2$  then  $\mathfrak{D}(L^\dagger) \in \mathcal{C}_0^c \tilde{\mathcal{B}}^2$ . If  $L \in \mathcal{C}^c \tilde{\mathcal{B}}^2$  then  $\mathfrak{D}(L^\dagger) \in \mathcal{C}^c \tilde{\mathcal{B}}^2$ .*

**5.3.** The bifunctor  $\mathcal{C}_0^c \tilde{\mathcal{B}}^2 \times \mathcal{C}_0^c \tilde{\mathcal{B}}^2 \rightarrow \mathcal{C}_0^c \tilde{\mathcal{B}}^2$ ,  $L, L' \mapsto L \underline{\circ} L'$  in 3.10 gives rise to a bifunctor  $\mathcal{C}^c \tilde{\mathcal{B}}^2 \times \mathcal{C}^c \tilde{\mathcal{B}}^2 \rightarrow \mathcal{C}^c \tilde{\mathcal{B}}^2$  denoted again by  $L, L' \mapsto L \underline{\circ} L'$  as follows. Let  $L \in \mathcal{C}^c \tilde{\mathcal{B}}^2$ ,  $L' \in \mathcal{C}^c \tilde{\mathcal{B}}^2$ ; by replacing if necessary  $\Psi$  by a power, we choose mixed structures of pure weight 0 on  $L, L'$ , we define  $L \underline{\circ} L'$  as in 3.10 in terms of these mixed structures and we then disregard the mixed structure on  $L \underline{\circ} L'$ . The resulting object of  $\mathcal{C}^c \tilde{\mathcal{B}}^2$  is denoted again by  $L \underline{\circ} L'$ ; it is independent of the choice of  $\Psi$  which defines the mixed structures.

Similarly for  $s, s'$  in  $\mathbf{Z}_c$ , the bifunctor  $\mathcal{C}_0^c Z_s \times \mathcal{C}_0^c Z_{s'} \rightarrow \mathcal{C}_0^c Z_{s+s'}$ ,  $L, L' \mapsto L \bullet L'$  in 4.24 gives rise to a bifunctor  $\mathcal{C}^c Z_s \times \mathcal{C}^c Z_{s'} \rightarrow \mathcal{C}^c Z_{s+s'}$  denoted again by  $L, L' \mapsto L \bullet L'$ . Moreover,  $\underline{\mathfrak{h}} : \mathcal{C}_0^c Z_s \rightarrow \mathcal{C}_0^c \tilde{\mathcal{B}}^2$  in 4.13 can be also viewed as a functor  $\underline{\mathfrak{h}} : \mathcal{C}^c Z_s \rightarrow \mathcal{C}^c \tilde{\mathcal{B}}^2$ .

The operation  $L \bullet L'$  (resp.  $L \underline{\circ} L'$ ) makes  $\sqcup_{s \in \mathbf{Z}_c} \mathcal{C}^c Z_s$  (resp.  $\mathcal{C}^c \tilde{\mathcal{B}}^2$ ) into a monoidal abelian category (see 4.24, 3.10). By [L16, 4.5(a)], we have:

(a) *For  $L, L'$  in  $\mathcal{C}^c \tilde{\mathcal{B}}^2$  we have canonically*

$$\mathrm{Hom}_{\mathcal{C}^c \tilde{\mathcal{B}}^2}(\mathbf{1}', L \underline{\circ} L') = \mathrm{Hom}_{\mathcal{C}^c \tilde{\mathcal{B}}^2}(\mathfrak{D}(L'^\dagger), L).$$

**5.4.** We set

$$(a) \quad \mathbf{1} = \oplus_{d \cdot \lambda \in \mathbf{D}_c} \beta_{d \cdot \lambda} \otimes \mathbf{L}_\lambda^{d-1} \in \mathcal{C}_0^c \tilde{\mathcal{B}}^2.$$

Here  $\beta_{d \cdot \lambda}$  is as in 5.1. By [L16, 4.7(g)],

(a)  $\mathbf{1} = \mathbf{1}'$  is a unit object of the monoidal category  $\mathcal{C}^c \tilde{\mathcal{B}}^2$ .

By [L16, 4.8], this monoidal category has a natural rigid structure.

**5.5.** In the remainder of this section we fix  $s \in \mathbf{Z}_c$ .

In this case,  $(\mathbf{e}^s)^*$  defines an equivalence of categories  $\mathcal{C}^c \tilde{\mathcal{B}}^2 \rightarrow \mathcal{C}^c \tilde{\mathcal{B}}^2$ ; this follows from 3.11(a).

By analogy with [L15, 6.2] and slightly extending a definition in [Mu, 3.1], we define an  $\mathbf{e}^s$ -half-braiding for an object  $\mathcal{L} \in \mathcal{C}^c \tilde{\mathcal{B}}^2$ , as a collection  $e_{\mathcal{L}} = \{e_{\mathcal{L}}(L); L \in \mathcal{C}^c \tilde{\mathcal{B}}^2\}$  where  $e_{\mathcal{L}}(L)$  is an isomorphism  $(\mathbf{e}^s)^*(L) \underline{\circ} \mathcal{L} \xrightarrow{\sim} \mathcal{L} \underline{\circ} L$  such that  $e_{\mathcal{L}}(\mathbf{1}) = \mathrm{Id}_{\mathcal{L}}$  and such that (i),(ii) below hold:

(i) If  $L \xrightarrow{t} L'$  is a morphism in  $\mathcal{C}^c \tilde{\mathcal{B}}^2$  then the diagram

$$\begin{array}{ccc} (\mathbf{e}^s)^*(L) \underline{\circ} \mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L)} & \mathcal{L} \underline{\circ} L \\ (\mathbf{e}^s)^*(t) \bullet \mathbf{1} \downarrow & & \mathbf{1} \bullet t \downarrow \\ (\mathbf{e}^s)^*(L') \underline{\circ} \mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L')} & \mathcal{L} \underline{\circ} L' \end{array}$$

is commutative.

(ii) If  $L, L' \in \mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2$  then  $e_{\mathcal{L}}(L \circ L') : (\mathbf{e}^s)^*(L \circ L') \circ \mathcal{L} \rightarrow \mathcal{L} \circ (L \circ L')$  is equal to the composition

$$(\mathbf{e}^s)^*(L) \circ (\mathbf{e}^s)^*(L') \circ \mathcal{L} \xrightarrow{1 \circ e_{\mathcal{L}}(L')} (\mathbf{e}^s)^*(L) \circ \mathcal{L} \circ L' \xrightarrow{e_{\mathcal{L}}(L) \circ 1} \mathcal{L} \circ L \circ L'.$$

(When  $s = 0$  this reduces to the definition of a half-braiding for  $\mathcal{L}$  given in [Mu, 3.1].)

Let  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$  be the category whose objects are the pairs  $(\mathcal{L}, e_{\mathcal{L}})$  where  $\mathcal{L}$  is an object of  $\mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2$  and  $e_{\mathcal{L}}$  is an  $\mathbf{e}^s$ -half-braiding for  $\mathcal{L}$ . For  $(\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'})$  in  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$  we define  $\text{Hom}_{\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}}((\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'}))$  to be the vector space consisting of all  $t \in \text{Hom}_{\mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2}(\mathcal{L}, \mathcal{L}')$  such that for any  $L \in \mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2$  the diagram

$$\begin{array}{ccc} (\mathbf{e}^s)^*(L) \circ \mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L)} & \mathcal{L} \circ L \\ 1 \circ t \downarrow & & t \circ 1 \downarrow \\ (\mathbf{e}^s)^*(L) \circ \mathcal{L}' & \xrightarrow{e_{\mathcal{L}'}(L')} & \mathcal{L}' \circ L \end{array}$$

is commutative. We say that  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$  is the  $\mathbf{e}^s$ -centre of  $\mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2$ . By a variation of a result of [Mu], [ENO] (which concerns the usual centre), the additive category  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$  is semisimple, with finitely many isomorphism classes of simple objects. By a variation of a general result on semisimple rigid monoidal categories in [ENO, Proposition 5.4], for any  $L \in \mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2$  one can define directly an  $\mathbf{e}^s$ -half-braiding on the object

$$\mathcal{I}_s(L) = \oplus_{y \cdot \lambda \in \mathbf{c}} (\mathbf{e}^s)^*(\mathbf{L}_{\lambda}^{\dot{y}}) \circ L \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} = \oplus_{y \cdot \lambda \in \mathbf{c}} \mathbf{L}_{\mathbf{e}^{-s}(y)}^{\mathbf{e}^{-s}(\lambda)} \circ L \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}$$

of  $\mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2$  such that, denoting by  $\overline{\mathcal{I}_s(L)}$  the corresponding object of  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$ , we have canonically

$$(a) \quad \text{Hom}_{\mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2}(L, L') = \text{Hom}_{\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}}(\overline{\mathcal{I}_s(L)}, L')$$

for any  $L' \in \mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$ . (We use that for  $y \cdot \lambda \in \mathbf{c}$ , the dual of the simple object  $\mathbf{L}_{\lambda}^{\dot{y}}$  is  $\mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}$ , see [L16, 4.4(c)]; we also use 3.11(a).) The  $\mathbf{e}^s$ -half-braiding on  $\mathcal{I}_s(L)$  can be described as follows: for any  $X \in \mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2$  we have canonically

$$\begin{aligned} (\mathbf{e}^s)^*(X) \circ \mathcal{I}_s(L) &= \oplus_{y \cdot \lambda \in \mathbf{c}} (\mathbf{e}^s)^*(X) \circ (\mathbf{e}^s)^*(\mathbf{L}_{\lambda}^{\dot{y}}) \circ L \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \\ &= \oplus_{y \cdot \lambda \in \mathbf{c}, z \cdot \lambda' \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2}((\mathbf{e}^s)^*(\mathbf{L}_{\lambda'}^{\dot{z}}), (\mathbf{e}^s)^*(X \circ \mathbf{L}_{\lambda}^{\dot{y}})) \otimes (\mathbf{e}^s)^*(\mathbf{L}_{\lambda'}^{\dot{z}}) \circ L \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \\ &= \oplus_{y \cdot \lambda \in \mathbf{c}, z \cdot \lambda' \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2}(\mathbf{L}_{\lambda'}^{\dot{z}}, X \circ \mathbf{L}_{\lambda}^{\dot{y}}) \otimes (\mathbf{e}^s)^*(\mathbf{L}_{\lambda'}^{\dot{z}}) \circ L \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \\ &= \oplus_{y \cdot \lambda \in \mathbf{c}, z \cdot \lambda' \in \mathbf{c}} \text{Hom}_{\mathcal{C}^{\mathbf{e}} \tilde{\mathcal{B}}^2}(\mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}, \mathbf{L}_{z(\lambda')}^{\dot{z}^{-1}} \otimes X) \otimes (\mathbf{e}^s)^*(\mathbf{L}_{\lambda'}^{\dot{z}}) \circ L \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}} \\ &= \oplus_{z \cdot \lambda' \in \mathbf{c}} (\mathbf{e}^s)^*(\mathbf{L}_{\lambda'}^{\dot{z}}) \circ L \circ \mathbf{L}_{z(\lambda')}^{\dot{z}^{-1}} \otimes X = \mathcal{I}_s(L) \circ X. \end{aligned}$$

(The fourth equality uses 4.20(a); we have also used 3.11(a).) We show:

(b) *If  $z \cdot \lambda \in \mathbf{c}$  and  $\mathcal{I}_s(\mathbf{L}_\lambda^\dot{z}) \neq 0$  then  $z \cdot \lambda \in \mathbf{c}^s$ .*

For some  $y \cdot \lambda' \in \mathbf{c}$  we have  $\mathbf{L}_{\mathbf{e}^{-s}(\lambda')}^{\mathbf{e}^{-s}(\dot{y})} \circ \mathbf{L}_\lambda^\dot{z} \neq 0$  (hence  $\mathbf{e}^{-s}(\lambda') = z(l)$ ) and  $\mathbf{L}_\lambda^\dot{z} \circ \mathbf{L}_{y(\lambda')}^{\dot{y}^{-1}} \neq 0$  (hence  $\lambda = \lambda'$ ). It follows that  $z(\lambda) = \mathbf{e}^{-s}(\lambda)$  and (b) is proved.

**5.6.** By 4.13(d), for  $z \cdot \lambda \in \mathbf{c}^s$  we have canonically

$$(a) \quad \underline{\mathbf{b}}(\mathbb{L}_{\lambda,s}^\dot{z}) = \mathcal{I}_s(\mathbf{L}_\lambda^\dot{z})$$

as objects of  $\mathcal{C}^c \tilde{\mathcal{B}}^2$ . Here  $\underline{\mathbf{b}} : \mathcal{C}^c Z_s \rightarrow \mathcal{C}^c \tilde{\mathcal{B}}^2$  is as in 5.3. Now  $\mathcal{I}_s(\mathbf{L}_\lambda^\dot{z})$  has a natural  $\mathbf{e}^s$ -half-braiding (by 5.5) and  $\underline{\mathbf{b}}(\mathbb{L}_{\lambda,s}^\dot{z})$  has a natural  $\mathbf{e}^s$ -half-braiding (by 4.14(j)). By 4.21(b),

(b) *these two  $\mathbf{e}^s$ -half-braidings are compatible with the identification (a).*

In view of (a),(b) we can reformulate 5.5(a) as follows.

**Theorem 5.7.** *For any  $z \cdot \lambda \in \mathbf{c}^s$ ,  $L' \in \mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$ , we have canonically*

$$(a) \quad \text{Hom}_{\mathcal{C}^c \tilde{\mathcal{B}}^2}(\mathbf{L}_\lambda^\dot{z}, L') = \text{Hom}_{\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}}(\overline{\underline{\mathbf{b}}(\mathbb{L}_{\lambda,s}^\dot{z})}, L')$$

where  $\overline{\underline{\mathbf{b}}(\mathbb{L}_{\lambda,s}^\dot{z})}$  is  $\underline{\mathbf{b}}(\mathbb{L}_{\lambda,s}^\dot{z})$  viewed as an object of  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$  with the  $\mathbf{e}^s$ -half-braiding given by 4.14(j).

**5.8.** We set

$$\mathbf{1}'_0 = \oplus_{d \cdot \lambda \in \mathbf{D}_c} \beta_{d \cdot \lambda}^* \otimes \mathbb{L}_{\lambda,0}^d \in \mathcal{C}^c Z_0.$$

From the definitions we have  $\tilde{\epsilon}_0 \mathbf{1}'_0 = \mathbf{1}'$ . Since  $\mathbf{1}' = \mathbf{1}$ , we have also  $\tilde{\epsilon}_0 \mathbf{1}'_0 = \mathbf{1}$ . We show:

(a) *For  $L \in \mathcal{C}^c Z_{-s}$ ,  $L' \in \mathcal{C}^c Z_s$  we have*

$$\text{Hom}_{\mathcal{M}(Z_0)}(\mathbf{1}'_0, L \bullet L') = \text{Hom}_{\mathcal{M}(Z_{-s})}(\mathfrak{D}(L'^\dagger), L).$$

We can assume that  $L = \mathbb{L}_{\lambda,-s}^{\dot{w}}$ ,  $L' = \mathbb{L}_{\lambda',s}^{\dot{w}'}$  where  $w \cdot \lambda \in \mathbf{c}^{-s}$ ,  $w' \cdot \lambda' \in \mathbf{c}^s$ . Using the fully faithfulness of  $\tilde{\epsilon}_0 : \mathcal{M}(Z_0) \rightarrow \mathcal{M} \tilde{\mathcal{B}}^2$ ,  $\tilde{\epsilon}_{-s} : \mathcal{M}(Z_{-s}) \rightarrow \mathcal{M} \tilde{\mathcal{B}}^2$ , and the equality  $\tilde{\epsilon}_0 \mathbf{1}'_0 = \mathbf{1}$ , we see that it is enough to prove that

$$\text{Hom}_{\mathcal{M}(\tilde{\mathcal{B}}^2)}(\mathbf{1}, \tilde{\epsilon}_0(L \bullet L')) = \text{Hom}_{\mathcal{M} \tilde{\mathcal{B}}^2}(\tilde{\epsilon}_{-s}(\mathfrak{D}(L'^\dagger)), \tilde{\epsilon}_{-s}(L)).$$

From 4.3 we have  $\tilde{\epsilon}_{-s}(L) = \mathbf{L}_\lambda^{\dot{w}}$ ,  $\tilde{\epsilon}_s(L') = \mathbf{L}_{\lambda'}^{\dot{w}'}$ ,  $\tilde{\epsilon}_{-s}(\mathbb{L}_{w'(\lambda'),-s}^{\dot{w}'^{-1}}) = \mathbf{L}_{w'(\lambda')}^{\dot{w}'^{-1}}$ .

From 4.3(e) we have

$$\tilde{\epsilon}_{-s}(\mathfrak{D}(L'^\dagger)) = \tilde{\epsilon}_{-s}(\mathfrak{D}(\mathbb{L}_{w'(\lambda')^{-1},-s}^{\dot{w}'^{-1}})) = \tilde{\epsilon}_{-s}(\mathbb{L}_{w'(\lambda')}^{\dot{w}'^{-1}}) = \mathbf{L}_{w'(\lambda')}^{\dot{w}'^{-1}}.$$

(We have use that  $\mathfrak{D}(\mathbb{L}_{w'(\lambda')^{-1},-s}^{\dot{w}'^{-1}}) = \mathbb{L}_{w'(\lambda'),-s}^{\dot{w}'^{-1}}$  which follows from [L16, 4.4(a)].)

Using 4.24(b), we have

$$\tilde{\epsilon}_0(L \bullet L') = (\tilde{\epsilon}_{-s} L) \circ (\tilde{\epsilon}_s L') = \mathbf{L}_\lambda^{\dot{w}} \circ \mathbf{L}_{\lambda'}^{\dot{w}'}$$

Hence it is enough to prove

$$\text{Hom}_{\mathcal{M} \tilde{\mathcal{B}}^2}(\mathbf{1}, \mathbf{L}_\lambda^{\dot{w}} \circ \mathbf{L}_{\lambda'}^{\dot{w}'}) = \text{Hom}_{\mathcal{M} \tilde{\mathcal{B}}^2}(\mathbf{L}_{w'(\lambda')}^{\dot{w}'^{-1}}, \mathbf{L}_\lambda^{\dot{w}}).$$

This follows from [L16, 4.5(a)].

6. TRUNCATED INDUCTION, TRUNCATED  
RESTRICTION, TRUNCATED CONVOLUTION

**6.1.** *In this section we fix  $s \in \mathbf{Z}$ .*

Let  $\dot{Z}_s = \{(B, B', \gamma) \in \mathcal{B} \times \mathcal{B} \times \tilde{G}_s; \gamma B \gamma^{-1} = B'\}$ . We have a diagram

$$(a) \quad Z_s \xleftarrow{f} \dot{Z}_s \xrightarrow{\pi} \tilde{G}_s$$

where  $f(B, B', \gamma) = (B, B', \gamma U_B)$ ,  $\pi(B, B', \gamma) = \gamma$ . Note that  $G$  acts on  $Z_s$  by  $g : (B, B', \gamma U_B) \mapsto (gBg^{-1}, gB'g^{-1}, g\gamma g^{-1}U_{gBg^{-1}})$ , on  $\dot{Z}_s$  by  $g : (B, B', \gamma) \mapsto (gBg^{-1}, gB'g^{-1}, g\gamma g^{-1})$ , on  $\tilde{G}_s$  by  $g : \gamma \mapsto g\gamma g^{-1}$ ; moreover,  $f$  and  $\pi$  are compatible with these  $G$ -actions. We define  $\chi : \mathcal{D}(Z_s) \rightarrow \mathcal{D}(\tilde{G}_s)$  by

$$\chi(L) = \pi_! f^* L.$$

For any  $w \cdot \lambda \in I$  we define  $\mathfrak{R}_{\lambda,s}^w \in \mathcal{D}(\tilde{G}_s)$ ,  $R_{\lambda,s}^w \in \mathcal{D}(\tilde{G}_s)$  by

$$\mathfrak{R}_{\lambda,s}^w = \chi(\mathcal{L}_{\lambda,s}^w), R_{\lambda,s}^w = \chi(\mathcal{L}_{\lambda,s}^{\sharp}), \text{ if } w \cdot \lambda \in I^s,$$

$$\mathfrak{R}_{\lambda}^w = 0, R_{\lambda}^w = 0 \text{ if } w \cdot \lambda \notin I^s.$$

Assume now that  $s \neq 0$  and that we are in case A. In this case, the conjugation  $G$ -action on  $\tilde{G}_s$  is transitive, see 2.1, and the stabilizer of  $\tau^s$  for this  $G$ -action is the finite group  $G^{\mathbf{e}^s} = \{g \in G; \mathbf{e}^s(g) = g\}$ .

With the notation of 4.1, for  $w \in W$  we have isomorphisms

$$X_s^w \xrightarrow{\sim} \pi^{-1}(\tau^s) \cap f^{-1}(Z_s^w), \bar{X}_s^w \xrightarrow{\sim} \pi^{-1}(\tau^s) \cap f^{-1}(\bar{Z}_s^w)$$

given by  $B \mapsto (B, \mathbf{e}^s(B), \tau^s)$ . Using this, and the transitivity of the  $G$ -action on  $\tilde{G}_s$ , we see that for  $w \cdot \lambda \in I^s$  and for  $j \in \mathbf{Z}$ ,  $(\mathfrak{R}_{\lambda,s}^w)^j[-\Delta]$  (resp.  $(R_{\lambda,s}^w)^j[-\Delta]$ ) is the  $G$ -equivariant local system on  $\tilde{G}_s$  whose stalk at  $\tau^s$  is  $H_c^{j-\Delta}(X_s^z, \mathcal{F}_{\lambda,s}^w)[\Delta]$  (resp.  $IH^{j-\Delta}(\bar{X}_s^z, \mathcal{F}_{\lambda,s}^w)[\Delta]$ ) with the  $G^{\mathbf{e}^s}$ -action considered in 4.1.

We return to the general case. We say that a simple perverse sheaf  $A$  on  $\tilde{G}_s$  is a *character sheaf* if the following equivalent conditions are satisfied:

- (i) there exists  $w \cdot \lambda \in I$  such that  $(A : \oplus_j (\mathfrak{R}_{\lambda,s}^w)^j) \neq 0$ ;
- (ii) there exists  $w \cdot \lambda \in I$  such that  $(A : (R_{\lambda,s}^w)^j) \neq 0$ .

In case A with  $s \neq 0$ , if  $A$  satisfies either (i) or (ii), then it must be  $G$ -equivariant, hence  $A[-D]$  must be a  $G$ -equivariant local system whose stalk at  $\tau^s$  viewed as a  $G^{\mathbf{e}^s}$ -module is irreducible, so that in this case the equivalence of (i),(ii) follows from the equivalence of (i),(ii) in 4.1. In case A with  $s = 0$  the equivalence of (i),(ii) follows from [L6, 12.7]; a similar proof applies in case B (see also [L9, 28.13]).

A character sheaf  $A$  determines a  $W$ -orbit  $\mathfrak{o}$  on  $\mathfrak{s}_{\infty}$ : the set of  $\lambda \in \mathfrak{s}_{\infty}$  such that  $(A : \oplus_j (\mathfrak{R}_{\lambda,s}^w)^j) \neq 0$  for some  $w \in W$  (or equivalently  $(A : \oplus_j (R_{\lambda,s}^w)^j) \neq 0$  for

some  $w \in W$ ); we have necessarily  $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$ . In case A with  $s \neq 0$  this follows from 4.1. In case A with  $s = 0$  this follows from [L6, 11.2(a), 12.7]; a similar proof applies in case B.

We now fix  $\mathfrak{o} \in W \backslash \mathfrak{s}_\infty$  such that  $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$ . We say that  $A$  is an  $\mathfrak{o}$ -character sheaf if the  $W$ -orbit on  $\mathfrak{s}_\infty$  determined by  $A$  is  $\mathfrak{o}$ . Let  $CS_{\mathfrak{o},s}$  be a set of representatives for the isomorphism classes of  $\mathfrak{o}$ -character sheaves on  $\tilde{G}_s$ . In case A with  $s \neq 0$  we have a natural bijection  $CS_{\mathfrak{o},s} \leftrightarrow \text{Irr}_{\mathfrak{o}}(G^{\mathbf{e}^s})$  (notation of 4.1); to  $A \in CS_{\mathfrak{o},s}$  corresponds the stalk of the  $G$ -equivariant local system  $A[-\Delta]$  at  $\tau^s$ , viewed as an irreducible  $G^{\mathbf{e}^s}$ -module.

Let  $\mathfrak{o} \in W \backslash \mathfrak{s}_\infty$  be such that  $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$ . With notation in 2.4 we have the following result.

(b) *There exists a pairing  $CS_{\mathfrak{o},s} \times \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1) \rightarrow \bar{\mathbf{Q}}_l$ ,  $(A, E) \mapsto b_{A,E}$  such that for any  $A \in CS_{\mathfrak{o},s}$ , any  $z \cdot \lambda \in I$  with  $\lambda \in \mathfrak{o}$  and any  $j \in \mathbf{Z}$  we have*

$$(A : (R_{\lambda,s}^z)^j) = (-1)^{j+\Delta}(j - \Delta - |z|; \sum_{E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)} b_{A,E} \text{tr}(\mathbf{e}_s c_{z \cdot \lambda}, E^v)).$$

Assume first that  $z \cdot \lambda \in I^s$ . In case A with  $s \neq 0$ , (b) follows from 4.1(b). In case A with  $s = 0$ , (b) is a reformulation of [L6, 14.11], see [L16, 5.1]. In case B, (b) can be deduced from [L10, 34.19] and the quasi-rationality result [L11, 39.8]. (In *loc.cit.* there is the assumption that the adjoint group of  $G$  is simple, which was made to simplify the arguments.)

Next we assume that  $z \cdot \lambda \in I - I^s$ . Then the left hand side of (a) is zero; hence it is enough to show that  $\text{tr}(\mathbf{e}_s c_{z \cdot \lambda}, E^v) = 0$  for any  $E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)$ . We have a direct sum decomposition  $E^v = \bigoplus_{\lambda' \in \mathfrak{s}_\infty} 1_{\lambda'} E^v$ . It is enough to show that for  $\lambda' \in \mathfrak{s}_\infty$  we have  $\mathbf{e}_s c_{z \cdot \lambda}(1_{\lambda'} E^v) \subset 1_{\lambda''} E^v$  where  $\lambda'' \in \mathfrak{s}_\infty$ ,  $\lambda'' \neq \lambda'$ . We can assume that  $\lambda' = \lambda$ . We have

$$\mathbf{e}_s c_{z \cdot \lambda}(1_{\lambda} E^v) \subset \mathbf{e}_s(1_{z(\lambda)} E^v) = 1_{\mathbf{e}^s(z(\lambda))} E^v.$$

It is enough to show that  $\mathbf{e}^s(z(\lambda)) \neq \lambda$  that is,  $z(\lambda) \neq \mathbf{e}^{-s}(\lambda)$ ; this follows from  $z \cdot \lambda \notin I^s$ .

Given  $A \in CS_{\mathfrak{o},s}$ , there is a unique two-sided cell  $\mathbf{c}_A$  of  $I$  such that  $b_{A,E} = 0$  whenever  $E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)$  satisfies  $\mathbf{c}_E \neq \mathbf{c}_A$ . In case A with  $s \neq 0$  this follows from results in [L1], under the assumption that the centre of  $G$  is connected; but the argument in [L1] extends to the general case. In case A with  $s = 0$  this follows from [L6, 16.7]. In case B this follows from [L12, §41]. We have necessarily  $\mathbf{c}_A \subset I_{\mathfrak{o}}$ . As in [L12, 41.8], [L13, 44.18], we see that:

(c) *We have  $(A : \bigoplus_j (R_{\lambda,s}^z)^j) \neq 0$  for some  $z \cdot \lambda \in \mathbf{c}_A$ ; conversely, if  $z \cdot \lambda \in I$  is such that  $(A : \bigoplus_j (R_{\lambda,s}^z)^j) \neq 0$ , then  $\mathbf{c}_A \preceq z \cdot \lambda$ .*

Let  $a_A$  be the value of the  $a$ -function on  $\mathbf{c}_A$ . If  $z \cdot \lambda \in I^s$ ,  $E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)$  satisfy  $\text{tr}(\mathbf{e}_s c_{z \cdot \lambda}, E^v) \neq 0$  then  $\mathbf{c}_E \preceq z \cdot \lambda$ ; if in addition we have  $z \cdot \lambda \in \mathbf{c}_E$  then from the definitions we have

$$\text{tr}(\mathbf{e}_s c_{z \cdot \lambda}, E^v) = \sum_{h \geq 0} c_{z \cdot \lambda, E, h, s} v^{a_E - h}$$

where  $c_{z \cdot \lambda, E, h, s} \in \bar{\mathbf{Q}}_l$  is zero for large  $h$ ,  $c_{z \cdot \lambda, E, 0, s} = \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty)$  and  $a_E$  is as in 1.13. Hence from (b) we see that for  $A \in CS_{\mathfrak{o}, s}$  and  $z \cdot \lambda \in I_{\mathfrak{o}}$ ,  $j \in \mathbf{Z}$ , the following holds:

(d) *We have  $(A : (R_{\lambda, s}^z)^j) = 0$  unless  $\mathbf{c}_A \preceq z \cdot \lambda$ ; if  $z \cdot \lambda \in \mathbf{c}_A$ , then*

$$(A : (R_{\lambda, s}^z)^j) = (-1)^{j+\Delta}(j - \Delta - |z|; \sum_{E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1); \mathbf{c}_E = \mathbf{c}_A; h \geq 0} b_{A, E} c_{z \cdot \lambda, E, h, s} v^{a_A - h})$$

which is 0 unless  $j - \Delta - |z| \leq a_A$ .

In the remainder of this section let  $\mathbf{c}, a, n, \Psi$  be as in 3.1(a). We assume that  $w \cdot \lambda \in \mathbf{c} \implies \lambda \in \mathfrak{o}$ .

Note that  $\chi$  can be also viewed as a functor  $\chi : \mathcal{D}_m(Z_s) \rightarrow \mathcal{D}_m(\tilde{G}_s)$ .

Let  $\mathcal{M}^{\preceq} \tilde{G}_s$  (resp.  $\mathcal{M}^{\prec} \tilde{G}_s$ ) be the category of perverse sheaves on  $\tilde{G}_s$  whose composition factors are all of the form  $A \in CS_{\mathfrak{o}, s}$  with  $\mathbf{c}_A \preceq \mathbf{c}$  (resp.  $\mathbf{c}_A \prec \mathbf{c}$ ). Let  $\mathcal{D}^{\preceq} \tilde{G}_s$  (resp.  $\mathcal{D}^{\prec} \tilde{G}_s$ ) be the subcategory of  $\mathcal{D}(\tilde{G}_s)$  whose objects are complexes  $K$  such that  $K^j$  is in  $\mathcal{M}^{\preceq} \tilde{G}_s$  (resp.  $\mathcal{M}^{\prec} \tilde{G}_s$ ) for any  $j$ . Let  $\mathcal{D}_m^{\preceq} \tilde{G}_s$  (resp.  $\mathcal{D}_m^{\prec} \tilde{G}_s$ ) be the subcategory of  $\mathcal{D}_m(\tilde{G}_s)$  whose objects are also in  $\mathcal{D}^{\preceq} \tilde{G}_s$  (resp.  $\mathcal{D}^{\prec} \tilde{G}_s$ ).

Let  $z \cdot \lambda \in I_{\mathfrak{o}}$ . From (d) we deduce:

- (e) *If  $z \cdot \lambda \preceq \mathbf{c}$ , then  $(R_{\lambda, s}^z)^j \in \mathcal{M}^{\preceq} \tilde{G}_s$  for all  $j \in \mathbf{Z}$ .*
- (f) *If  $z \cdot \lambda \in \mathbf{c}$  and  $j > a + \Delta + |z|$  then  $(R_{\lambda, s}^z)^j \in \mathcal{M}^{\prec} \tilde{G}_s$ .*
- (g) *If  $z \cdot \lambda \prec \mathbf{c}$  then  $(R_{\lambda, s}^z)^j \in \mathcal{M}^{\prec} \tilde{G}_s$  for all  $j \in \mathbf{Z}$ .*

**6.2.** Let  $CS_{\mathbf{c}, s} = \{A \in CS_{\mathfrak{o}, s}; \mathbf{c}_A = \mathbf{c}\}$ . For any  $z \cdot \lambda \in I$  we set

$$n_z = a(z) + \Delta + |z|.$$

Let  $A \in CS_{\mathbf{c}, s}$  and let  $z \cdot \lambda \in \mathbf{c}$ . We have

$$(a) \quad (A : (R_{\lambda, s}^z)^{n_z}) = (-1)^{a+|z|} \sum_{E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)} b_{A, E} \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty).$$

Indeed, from 6.1(b) we have

$$(A : (R_{\lambda, s}^z)^{n_z}) = (-1)^{a+|z|} \sum_{E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)} b_{A, E} (a; \text{tr}(\mathbf{e}_s c_{z \cdot \lambda}, E^v))$$

and it remains to use that  $(a; \text{tr}(\mathbf{e}_s c_{z \cdot \lambda}, E^v)) = \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty)$ . We show:

(b) *For any  $A \in CS_{\mathbf{c}, s}$  there exists  $E \in \text{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)$  such that  $b_{A, E} \neq 0$  hence  $\mathbf{c}_E = \mathbf{c}$ .*

Assume that this is not so. Then, using 6.1(b), for any  $z \cdot \lambda \in I_{\mathfrak{o}}$  we have  $(A : \oplus_j (R_{\lambda, s}^z)^j) = 0$ . This contradicts the assumption that  $A \in CS_{\mathfrak{o}, s}$ . We show:

(c) For any  $A \in CS_{\mathbf{c},s}$  there exists  $z \cdot \lambda \in \mathbf{c}$  such that  $(A : (R_{\lambda,s}^z)^{n_z}) \neq 0$ . Assume that this is not so. Then, using (a), we see that

$$\sum_{E \in \text{Irr}_s(\mathbf{H}_o^1); \mathbf{c}_E = \mathbf{c}} b_{A,E} \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty) = 0$$

for any  $z \cdot \lambda \in \mathbf{c}$ . If  $z \cdot \lambda \in I_o - \mathbf{c}$  then the last sum is automatically zero since  $t_{z \cdot \lambda}$  acts as 0 on  $E^\infty$  for each  $E$  in the sum. Thus we have

$$\sum_{E \in \text{Irr}_s(\mathbf{H}_o^1); \mathbf{c}_E = \mathbf{c}} b_{A,E} \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty) = 0$$

for any  $z \cdot \lambda \in I_o$ . In the last sum the condition  $\mathbf{c}_E = \mathbf{c}$  is automatically satisfied if  $b_{A,E} \neq 0$ . Thus we have

$$\sum_{E \in \text{Irr}_s(\mathbf{H}_o^1)} b_{A,E} \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty) = 0$$

for any  $z \cdot \lambda \in I_o$ . By a general argument (see for example [L10, 34.14(e)]), the linear functions  $t_{z \cdot \lambda} \mapsto \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty)$ ,  $\mathbf{J}_o \rightarrow \bar{\mathbf{Q}}_l$  (for various  $E$  as in the last sum) are linearly independent. It follows that  $b_{A,E} = 0$  for each  $E$  as in the last sum. This contradicts (b).

We show:

(d) Let  $z \cdot \lambda \in \mathbf{c}$  be such that  $(R_{\lambda,s}^z)^{n_z} \neq 0$ . Then  $z \cdot \lambda \underset{\text{left}}{\sim} \mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda))$  and  $z \cdot \lambda \underset{\text{left}}{\sim} \mathbf{e}^s(z^{-1}) \cdot \lambda$ .

Using (a) we see that there exists  $E \in \text{Irr}_s(\mathbf{H}_o^1)$  such that  $\text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty) \neq 0$ . We have  $E^\infty = \bigoplus_{d \cdot \lambda_1 \in \mathbf{D} \cap \mathfrak{o}} t_{d \cdot \lambda_1} E^\infty$ . We define  $d \cdot \lambda_1 \in \mathbf{D} \cap \mathfrak{o}$  by the condition that  $z \cdot \lambda \underset{\text{left}}{\sim} d \cdot \lambda_1$ . We define  $d' \cdot \lambda'_1 \in \mathbf{D} \cap \mathfrak{o}$  by the condition that  $z^{-1} \cdot z(\lambda) \underset{\text{left}}{\sim} d' \cdot \lambda'_1$ . Now  $t_{z \cdot \lambda} : E^\infty \rightarrow E^\infty$  maps the summand  $t_{d \cdot \lambda_1} E^\infty$  into the summand  $t_{d' \cdot \lambda'_1} E^\infty$  and all other summands to zero. Moreover,  $\mathbf{e}_s$  maps  $t_{d' \cdot \lambda'_1} E^\infty$  into  $t_{\mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1)} E^\infty$ . Hence  $\mathbf{e}_s t_{z \cdot \lambda} : E^\infty \rightarrow E^\infty$  maps the summand  $t_{d \cdot \lambda_1} E^\infty$  into the summand  $t_{\mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1)} E^\infty$  and all other summands to zero. Since  $\text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty) \neq 0$  it follows that  $t_{d \cdot \lambda_1} E^\infty = t_{\mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1)} E^\infty \neq 0$ . Since  $\mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1) \in \mathbf{D} \cap \mathfrak{o}$ , it follows that  $d \cdot \lambda_1 = \mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1)$ . Since  $\mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda)) \underset{\text{left}}{\sim} \mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1)$ , we see that  $z \cdot \lambda \underset{\text{left}}{\sim} \mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda))$ . To complete the proof, it remains to note that  $\mathbf{e}^s(z(\lambda)) = \lambda$  that is  $z \cdot \lambda \in I^s$ . This follows from the fact that  $(R_{\lambda,s}^z)^{n_z} \neq 0$ .

We show:

(e) If  $CS_{\mathbf{c},s} \neq \emptyset$  then  $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$ .

Using (c) and the hypothesis we see that there exists  $z \cdot \lambda \in \mathbf{c}$  such that  $(R_{\lambda,s}^z)^{n_z} \neq 0$ . Using (d), we see that  $\mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda)) \in \mathbf{c}$ . Since  $z^{-1} \cdot z(\lambda) \in \mathbf{c}$  (see Q10 in 1.9) we have also  $\mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda)) \in \mathbf{e}^s(\mathbf{c})$ . Thus,  $\mathbf{c} \cap \mathbf{e}^s(\mathbf{c}) \neq \emptyset$ . It follows that  $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$ .



**6.3.** *Until the end of 6.7 we assume that  $s \in \mathbf{Z}_c$ .*

We show:

- (a) *If  $L \in \mathcal{D}^\preceq Z_s$  then  $\chi(L) \in \mathcal{D}^\preceq \tilde{G}_s$ . If  $L \in \mathcal{D}^\succ Z_s$  then  $\chi(L) \in \mathcal{D}^\succ \tilde{G}_s$ .*
- (b) *If  $L \in \mathcal{M}^\preceq Z_s$  and  $j > a + \nu$  then  $(\chi(L))^j \in \mathcal{M}^\succ \tilde{G}_s$ .*

It is enough to prove (a),(b) assuming in addition that  $L = \mathbb{L}_{\lambda,z}^{\dot{z}}$  where  $z \cdot \lambda \in I^s$ ,  $z \cdot \lambda \preceq c$ . Then (a) follows from 6.1(e),(g). In the setup of (b) we have

$$(\chi(\mathbb{L}_{\lambda,s}^{\dot{z}}))^j = (R_{\lambda}^{\dot{z}})^{j+|z|+\nu+\rho}((|z| + \nu + \rho)/2)$$

and this is in  $\mathcal{M}^\succ G$  since  $j + |z| + \nu + \rho > a + \Delta + |z|$ , see 6.1(f).

**6.4.** Let  $\mathcal{C}^\spadesuit \tilde{G}_s$  be the subcategory of  $\mathcal{M}(\tilde{G}_s)$  consisting of semisimple objects. Let  $\mathcal{C}_0^\spadesuit \tilde{G}_s$  be the subcategory of  $\mathcal{M}_m(\tilde{G}_s)$  consisting of objects of pure of weight zero. Let  $\mathcal{C}^c \tilde{G}_s$  be the subcategory of  $\mathcal{M}(\tilde{G}_s)$  consisting of objects which are direct sums of objects in  $CS_{c,s}$ . Let  $\mathcal{C}_0^c \tilde{G}_s$  be the subcategory of  $\mathcal{C}_0^\spadesuit \tilde{G}_s$  consisting of those  $K$  such that, as an object of  $\mathcal{C}^\spadesuit \tilde{G}_s$ ,  $K$  belongs to  $\mathcal{C}^c \tilde{G}_s$ . For  $K \in \mathcal{C}_0^\spadesuit \tilde{G}_s$  let  $\underline{K}$  be the largest subobject of  $K$  such that as an object of  $\mathcal{C}^\spadesuit \tilde{G}_s$ , we have  $\underline{K} \in \mathcal{C}^c \tilde{G}_s$ .

**6.5.** For  $L \in \mathcal{C}_0^c Z_s$  we set

$$\underline{\chi}(L) = \underline{(\chi(L))^{a+\nu}}((a+\nu)/2) = \underline{(\chi(L))^{\{a+\nu\}}} \in \mathcal{C}_0^c \tilde{G}_s.$$

(The last equality uses that  $\pi$  in 6.1 is proper hence it preserves purity.) The functor  $\underline{\chi} : \mathcal{C}_0^c Z_s \rightarrow \mathcal{C}_0^c \tilde{G}_s$  is called *truncated induction*. For  $z \cdot \lambda \in c^s$  we have

$$(a) \quad \underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = \underline{(R_{\lambda,s}^{\dot{z}})^{n_z}}(n_z/2).$$

Indeed,

$$\begin{aligned} \underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}) &= \underline{(\chi(\mathbb{L}_{\lambda,s}^{\dot{z}}))^{a+\nu}}((a+\nu)/2) = \underline{(\chi(\mathcal{L}_{\lambda,s}^{\dot{z}} \langle |z| + \nu + \rho \rangle))^{a+\nu}}((a+\nu)/2) \\ &= \underline{(\chi(\mathcal{L}_{\lambda,s}^{\dot{z}}))^{|z|+a+\Delta}}((|z| + a + \Delta)/2) = \underline{(\chi(\mathcal{L}_{\lambda,s}^{\dot{z}}))^{n_z}}(n_z/2) = \underline{(R_{\lambda,s}^{\dot{z}})^{n_z}}(n_z/2). \end{aligned}$$

Using (a) and 6.2(d) we see that:

$$(d) \text{ If } z \cdot \lambda \in c^s \text{ is such that } \underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}) \neq 0 \text{ then } z \cdot \lambda \underset{\text{left}}{\sim} e^s(z^{-1}) \cdot \lambda.$$

**6.6.** For  $z \cdot \lambda, z' \cdot \lambda'$  in  $c^s$  we show:

$$(a) \quad \dim \text{Hom}_{\mathcal{C}^c \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{z}'})) = \sum_{u \cdot \lambda_1 \in c} \mathbf{t}(t_{u^{-1} \cdot u(\lambda_1)} t_{z \cdot \lambda} t_{e^s(u) \cdot e^s(\lambda_1)} t_{z'^{-1} \cdot z'(\lambda')})$$

where  $\mathbf{t} : \mathbf{H}^\infty \rightarrow \mathbf{Z}$  is as in 1.9.

Let  $(\cdot)^\spadesuit : \bar{\mathbf{Q}}_l \rightarrow \bar{\mathbf{Q}}_l$  be a field automorphism which maps any root of 1 in  $\bar{\mathbf{Q}}_l$  to its inverse. The field automorphism  $\bar{\mathbf{Q}}_l(v) \rightarrow \bar{\mathbf{Q}}_l(v)$  which maps  $v$  to  $v$  and  $x \in \bar{\mathbf{Q}}_l$  to  $x^\spadesuit$  is denoted again by  $\spadesuit$ .

Let  $N_1$  (resp.  $N_2$ ) be the left (resp. right) hand side of (a). Using 6.5(a) and the definitions we see that

$$(b) \quad N_1 = \sum_{A \in CS_{\mathbf{c},s}} (A : (R_{\lambda,s}^{\dot{z}})^{n_z}) (A : (R_{\lambda',s}^{\dot{z}'})^{n_{z'}}).$$

Using 6.2(a) and the analogous identity for  $(A : (R_{\lambda',s}^{\dot{z}'})^{n_{z'}})$  in which the field automorphism  $(\spadesuit) : \bar{\mathbf{Q}}_l \rightarrow \bar{\mathbf{Q}}_l$  is applied to both sides (the left hand side is fixed by  $(\spadesuit)$ ), we deduce that

$$N_1 = (-1)^{|z|+|z'|} \sum_{E, E' \in \text{Irr}_s(\mathbf{H}_0^1)} \sum_{A \in CS_{\mathbf{c},s}} b_{A,E} b_{A,E'}^{\spadesuit} \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty) \text{tr}(\mathbf{e}_s t_{z' \cdot \lambda'}, E'^\infty)^{\spadesuit}.$$

In the last sum we replace  $\sum_{A \in CS_{\mathbf{c},s}} b_{A,E} b_{A,E'}^{\spadesuit}$  by 1 if  $E' = E$  and by 0 if  $E' \neq E$ . (In case A with  $s \neq 0$  we use [L1, 3.9(i)] which assumes that the centre of  $G$  is connected, but a similar proof applies without assumption on the centre. In case A with  $s = 0$  and in case B we use [L10, 35.18(g)].)

We see that

$$N_1 = (-1)^{|z|+|z'|} \sum_{E \in \text{Irr}_s(\mathbf{H}_0^1)} \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty) \text{tr}(\mathbf{e}_s t_{z' \cdot \lambda'}, E^\infty)^{\spadesuit}.$$

We now use the equality (for  $E \in \text{Irr}_s(\mathbf{H}_0^1)$ ):

$$\text{tr}(\mathbf{e}_s t_{z' \cdot \lambda'}, E^\infty)^{\spadesuit} = \text{tr}(t_{z'^{-1} \cdot z'(\lambda')} \mathbf{e}_s^{-1}, E^\infty)$$

which can be deduced from [L10, 34.17]. We see that

$$N_1 = (-1)^{|z|+|z'|} \sum_{E \in \text{Irr}_s(\mathbf{H}_0^1)} \text{tr}(\mathbf{e}_s t_{z \cdot \lambda}, E^\infty) \text{tr}(t_{z'^{-1} \cdot z'(\lambda')} \mathbf{e}_s^{-1}, E^\infty).$$

This is equal to  $(-1)^{|z|+|z'|}$  times the trace of the linear map  $\xi \mapsto t_{z \cdot \lambda} \mathbf{e}^s(\xi) t_{z'^{-1} \cdot z'(\lambda')}$  from  $\mathbf{J}_0$  to  $\mathbf{J}_0$ ; hence it is equal to

$$(-1)^{|z|+|z'|} \sum_{u \cdot \lambda_1 \in \mathfrak{o}} \mathbf{t}(t_{u^{-1} \cdot u(\lambda_1)} t_{z \cdot \lambda} t_{\mathbf{e}^s(u) \cdot \mathbf{e}^s(\lambda_1)} t_{z'^{-1} \cdot z'(\lambda')}) = (-1)^{|z|+|z'|} N_2.$$

(In the last sum, the terms with  $u \cdot \lambda_1 \in \mathfrak{o} - \mathbf{c}$  contribute 0.) Thus,  $N_1 = (-1)^{|z|+|z'|} N_2$ . Since  $N_1$  and  $N_2$  are natural numbers it follows that  $N_1 = N_2$ . This proves (a).

The proof above shows also that  $\dim \text{Hom}_{C^c \tilde{G}_s} (\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{z}'})) = 0$  whenever  $(-1)^{|z|+|z'|} = -1$ .

Replacing in (a)  $u \cdot \lambda_1$  by  $\mathbf{e}^{-s}(y) \cdot \mathbf{e}^{-s}\lambda_1$  (recall that  $\mathbf{e}^s : \mathbf{c} \rightarrow \mathbf{c}$  is a bijection) we can rewrite (a) as follows:

$$\dim \operatorname{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{z}'})) = \sum_{y \cdot \lambda_1 \in \mathbf{c}} \mathbf{t}(t_{\epsilon^{-s}(y^{-1}) \cdot \mathbf{e}^{-s}(y(\lambda_1))} t_{z \cdot \lambda} t_{y \cdot \lambda_1} t_{z'^{-1}, z'(\lambda')}).$$

Since  $N_1$  (in the form (b)) is symmetric in  $z \cdot \lambda, z' \cdot \lambda'$ , we have also

$$\dim \operatorname{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{z}'})) = \sum_{y \cdot \lambda_1 \in \mathbf{c}} \mathbf{t}(t_{\epsilon^{-s}(y^{-1}) \cdot \mathbf{e}^{-s}(y(\lambda_1))} t_{z' \cdot \lambda'} t_{y \cdot \lambda_1} t_{z^{-1}, z(\lambda)}).$$

Replacing  $y \cdot \lambda_1$  by  $y^{-1} \cdot y(\lambda_1)$  (recall that  $y \cdot \lambda_1 \mapsto y^{-1} \cdot y(\lambda_1)$  is an involution  $\mathbf{c} \rightarrow \mathbf{c}$ ) we can rewrite this as follows:

$$(c) \quad \dim \operatorname{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{z}'})) = \sum_{y \cdot \lambda_1 \in \mathbf{c}} \mathbf{t}(t_{\epsilon^{-s}(y) \cdot \mathbf{e}^{-s}(\lambda_1)} t_{z' \cdot \lambda'} t_{y^{-1} \cdot y(\lambda_1)} t_{z^{-1}, z(\lambda)}).$$

We show:

(d) *There exist  $z \cdot \lambda \in \mathbf{c}^s$  such that  $\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}) \neq 0$ .*

Let  $k = u \cdot \lambda_1 \in \mathbf{c}$ . Then  $\mathbf{e}^s(k) \in \mathbf{c}$ ,  $k^! \in \mathbf{c}$  hence by 1.15(d) we have  $t_{k^!} t_j t_{\mathbf{e}^s(k)} \neq 0$  for some  $j \in I$ . From 2.5(a) we deduce that  $j \in \mathbf{c}^s$ . We can find  $j' = z' \cdot \lambda' \in \mathbf{c}$  such that  $t_{j'}$  appears with nonzero coefficient in  $t_{k^!} t_j t_{\mathbf{e}^s(k)}$ . It follows that  $\mathbf{t}(t_{k^!} t_j t_{\mathbf{e}^s(k)} t_{j'^!}) \neq 0$ . Since  $\mathbf{t}(\xi \xi') = \mathbf{t}(\xi' \xi)$  for  $\xi, \xi' \in \mathbf{H}^\infty$  we deduce that  $\mathbf{t}(t_{\mathbf{e}^s(k)} t_{j'^!} t_{k^!} t_j) \neq 0$ . In particular we have  $t_{\mathbf{e}^s(k)} t_{j'^!} t_{k^!} \neq 0$ . Applying the anti-automorphism  $t_u \mapsto t_{u^!}$  of  $\mathbf{H}^\infty$  we deduce  $t_k t_{j'} t_{\mathbf{e}^s(k^!)} \neq 0$ . Using again 2.5(a) we deduce that  $j' \in \mathbf{c}^s$ . If  $i \in \mathbf{c}$ ,  $j \in I$  satisfy  $t_i t_j t_{\mathbf{e}^s(i)} \neq 0$  then  $j \in \mathbf{c}^s$ . Since  $\mathbf{t}(t_h t_j t_{\mathbf{e}^s(h)} t_{j'^!}) \in \mathbf{N}$  for any  $h \in \mathbf{c}$  and  $\mathbf{t}(t_{k^!} t_j t_{\mathbf{e}^s(k)} t_{j'^!}) \neq 0$ , we see that  $\sum_{h \in \mathbf{c}} \mathbf{t}(t_h t_j t_{\mathbf{e}^s(h)} t_{j'^!}) \in \mathbf{N}_{>0}$ . Using this and (a), we see that

$$\dim \operatorname{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{z}'})) \in \mathbf{N}_{>0}.$$

This proves (d).

The following converse to 6.2(e) is an immediate consequence of (d):

(e) *We have  $CS_{\mathbf{c},s} \neq \emptyset$ .*

**6.7.** Let  $L \in \mathcal{C}_0^{\mathbf{c}} Z_s$ . We show that  $\mathfrak{D}(L) \in \mathcal{C}_0^{\tilde{\mathbf{c}}} Z_s$ . (Here  $\tilde{\mathbf{c}}$  is as in 1.14.) It is enough to note that for  $w \cdot \lambda \in \mathbf{c}^s$  and  $\omega \in \kappa_0^{-1}(w)$  we have

$$(a) \quad \mathfrak{D}(\mathbb{L}_{\lambda,s}^\omega) = \mathbb{L}_{\lambda^{-1},s}^\omega.$$

We show:

(b) *For  $L \in \mathcal{C}_0^{\mathbf{c}} Z_s$  we have canonically  $\underline{\chi}(\mathfrak{D}(L)) = \mathfrak{D}(\underline{\chi}(L))$  where the first  $\underline{\chi}$  is relative to  $\tilde{\mathbf{c}}$  instead of  $\mathbf{c}$ .*

Let  $\pi, f, \dot{Z}_s$  be as in 6.1. By the relative hard Lefschetz theorem [BBD, 5.4.10]

applied to the projective morphism  $\pi$  and to  $f^*L\langle\nu\rangle$  (a perverse sheaf of pure weight 0 on  $\dot{Z}_s$ ) we have canonically for any  $j \in \mathbf{Z}$ :

$$(c) \quad (\pi_! f^* L\langle\nu\rangle)^{-j} = (\pi_! f^* L\langle\nu\rangle)^j(j).$$

We have used the fact that  $f$  is smooth with fibres of dimension  $\nu$ . This also shows that

$$(d) \quad \mathfrak{D}(\chi(\mathfrak{D}(L))) = \chi(L)\langle 2\nu\rangle.$$

Using (d) we have

$$\begin{aligned} \mathfrak{D}(\underline{\chi}(\mathfrak{D}(L))) &= \mathfrak{D}((\chi(\mathfrak{D}(L)))^{a+\nu}((a+\nu)/2)) = (\mathfrak{D}(\chi(\mathfrak{D}(L))))^{-a-\nu}((-a-\nu)/2) \\ &= (\chi(L)\langle 2\nu\rangle)^{-a-\nu}((-a-\nu)/2) = (\chi(L)\langle\nu\rangle)^{-a}(-a/2). \end{aligned}$$

Hence using (c) we have

$$\mathfrak{D}(\underline{\chi}(\mathfrak{D}(L))) = (\chi(L)\langle\nu\rangle)^a(a/2) = (\chi(L))^{a+\nu}((a+\nu)/2) = \underline{\chi}(L).$$

This proves (b).

**6.8.** We define  $\zeta : \mathcal{D}(\tilde{G}_s) \rightarrow \mathcal{D}(Z_s)$  and  $\zeta : \mathcal{D}_m(\tilde{G}_s) \rightarrow \mathcal{D}_m(Z_s)$  by  $\zeta(K) = f_! \pi^* K$  where  $Z_s \xleftarrow{f} \dot{Z}_s \xrightarrow{\pi} \tilde{G}_s$  is as in 6.1(a). We show:

(a) *For any  $L \in \mathcal{D}(Z_s)$  or  $L \in \mathcal{D}_m(Z_s)$  we have  $\mathfrak{b}''(L) = \zeta(\chi(L))$ .*

We have  $\zeta(\chi(L)) = f_! \pi^* \pi_! f^*(L)$ . We have

$$\dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s = \{((B_0, B_1, B_2, B_3), \gamma) \in \mathcal{B}^4 \times \tilde{G}_s; \gamma B_0 \gamma^{-1} = B_3, \tilde{g} B_1 \tilde{g}^{-1} = B_2\}.$$

We have a cartesian diagram

$$\begin{array}{ccc} \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s & \xrightarrow{\tilde{\pi}_1} & \dot{Z}_s \\ \tilde{\pi}_2 \downarrow & & \downarrow \pi \\ \dot{Z}_s & \xrightarrow{\pi} & \tilde{G}_s \end{array}$$

where  $\tilde{\pi}_1((B_0, B_1, B_2, B_3), \gamma) = (B_0, B_3, \gamma)$ ,  $\tilde{\pi}_2((B_0, B_1, B_2, B_3), \gamma) = (B_1, B_2, \gamma)$ . It follows that  $\pi^* \pi_! = \tilde{\pi}_1! \tilde{\pi}_2^*$ . Thus,

$$\zeta(\chi(L)) = f_! \tilde{\pi}_1! \tilde{\pi}_2^* f^*(L) = (f \tilde{\pi}_1)_! (f \tilde{\pi}_2)^*(L).$$

Define  $\pi'_1 : \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s \rightarrow Z_s$ ,  $\pi'_2 : \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s \rightarrow Z_s$  by

$$\begin{aligned} \pi'_1((B_0, B_1, B_2, B_3), \gamma) &= (B_0, B_3, \gamma U_{B_0}), \\ \pi'_2((B_0, B_1, B_2, B_3), \gamma) &= (B_1, B_2, \gamma U_{B_1}). \end{aligned}$$

Then  $\pi'_1 = f \tilde{\pi}_1$ ,  $\pi'_2 = f \tilde{\pi}_2$  and  $\zeta(\chi(L)) = \pi'_{1!} \pi'^*_2(L)$ . Let  ${}^\diamond \mathcal{Y}$  be as in 4.14. We have an isomorphism  ${}^\diamond \mathcal{Y} \rightarrow \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s$  induced by

$$((x_0 \mathbf{U}, x_1 \mathbf{U}, x_2 \mathbf{U}, x_3 \mathbf{U}), \gamma) \mapsto ((x_0 \mathbf{B} x_0^{-1}, x_1 \mathbf{B} x_1^{-1}, x_2 \mathbf{B} x_2^{-1}, x_3 \mathbf{B} x_3^{-1}), \gamma).$$

We use this to identify  ${}^\diamond \mathcal{Y} = \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s$ . Then  $\pi'_1, \pi'_2$  become  $d, {}^\diamond \eta$  of 4.25. We see that (a) holds.

**6.9.** In the remainder of this section we assume that  $s \in \mathbf{Z}_{\mathbf{c}}$ .

Let  $z \cdot \lambda \in \mathfrak{o}$ . We set  $\Sigma = \epsilon_s^* \zeta(R_{\lambda,s}^z) \langle 2\nu + |z| \rangle \in \mathcal{D}(\tilde{\mathcal{B}}^2)$ . Let  $j \in \mathbf{Z}$ . We show:

- (a) If  $z \cdot \lambda \preceq \mathbf{c}$ , then  $\Sigma^j \in \mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$ .
- (b) If  $z \cdot \lambda \prec \mathbf{c}$ , then  $\Sigma^j \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ .
- (c) If  $z \cdot \lambda \in \mathbf{c}$  and  $j > \nu + 2\rho + 2a$ , then  $\Sigma^j \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ .

If  $z \cdot \lambda \notin I^s$ , then  $\Sigma = 0$  and there is nothing to prove. Now assume that  $z \cdot \lambda \in I^s$ . Using 4.9(a), we have

$$\Sigma = \epsilon_s^* \zeta(\chi(\mathcal{L}_{\lambda,s}^{\ddagger\#})) \langle 2\nu + |z| \rangle = \mathfrak{b}'(\mathcal{L}_{\lambda,s}^{\ddagger\#}) \langle 2\nu + |z| \rangle = \mathfrak{b}'(\mathbb{L}_{\lambda,s}^{\ddagger}) \langle \nu - \rho \rangle.$$

Now (a),(b) follow from 4.14(a),(b); (c) follows from 4.14(c). (If  $j > \nu + 2\rho + 2a$ , then  $j + \nu - r > 2\nu + \rho + 2a$ .)

**6.10.** We show:

- (a) If  $K \in \mathcal{D}^{\preceq} \tilde{G}_s$ , then  $\zeta(K) \in \mathcal{D}^{\preceq} Z_s$ .
- (b) If  $K \in \mathcal{D}^{\prec} \tilde{G}_s$ , then  $\zeta(K) \in \mathcal{D}^{\prec} Z_s$ .
- (c) If  $K \in \mathcal{D}^{\preceq} \tilde{G}_s$  and  $j > \nu + a$ , then  $(\zeta(K))^j \in \mathcal{M}^{\prec} Z_s$ .

We can assume in addition that  $K = A \in CS_{\mathbf{c}',s}$  for a two-sided cell  $\mathbf{c}'$  such that  $\mathbf{c}' \preceq \mathbf{c}$ . Assume first that  $\mathbf{c}' = \mathbf{c}$ . By 6.2(c) we can find  $z \cdot \lambda \in \mathbf{c}$  such that  $(A : (R_{\lambda,s}^z)^{n_z}) \neq 0$ . Then  $A[-n_z]$  (without mixed structure) is a direct summand of the semisimple complex  $R_{\lambda,s}^z$ . Hence  $\epsilon_s^* \zeta(A)[-n_z]$  is a direct summand of  $\epsilon_s^* \zeta(R_{\lambda,s}^z)$  and  $\epsilon_s^* \zeta(A)[-n_z + 2\nu + |z|]$  is a direct summand of  $\Sigma$  (in 6.9), that is,  $\epsilon_s^* \zeta(A)[-a - \rho]$  is a direct summand of  $\Sigma$ . By 6.9, if  $j \in \mathbf{Z}$  (resp.  $j > \nu + 2\rho + 2a$ ) then  $\Sigma^j \in \mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$  (resp.  $\Sigma^j \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ ) hence  $(\epsilon_s^* \zeta(A)[-a - \rho])^j \in \mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$  (resp.  $(\epsilon_s^* \zeta(A)[-a - \rho])^j \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ ), that is,  $(\epsilon_s^* \zeta(A))^{j-a-\rho} \in \mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$  (resp.  $(\epsilon_s^* \zeta(A))^{j-a-\rho} \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ ). We see that if  $j' \in \mathbf{Z}$  (resp.  $j' > \nu + \rho + a$ ) then  $(\epsilon_s^* \zeta(A))^{j'} \in \mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$  (resp.  $(\epsilon_s^* \zeta(A))^{j'} \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$ ), so that  $(\zeta(A))^{j'-\rho} \in \mathcal{M}^{\preceq} Z_s$  (resp.  $(\zeta(A))^{j'-\rho} \in \mathcal{M}^{\prec} Z_s$ ); here we use 4.3(a). We see that if  $j \in \mathbf{Z}$  (resp.  $j > \nu + a$ , so that  $j + \rho > \nu + \rho + a$ ), then  $(\zeta(A))^j \in \mathcal{M}^{\preceq} Z_s$  (resp.  $(\zeta(A))^j \in \mathcal{M}^{\prec} Z_s$ ). Thus the desired results hold when  $\mathbf{c}' = \mathbf{c}$ .

Assume now that  $\mathbf{c}' \prec \mathbf{c}$ . Applying the above argument with  $\mathbf{c}$  replaced by  $\mathbf{c}'$ , we see that the desired results hold.

**6.11.** For  $K \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_s$  we set

$$\underline{\zeta}(K) = \underline{(\zeta(K))}^{\{\nu+a\}} \in \mathcal{C}_0^{\mathbf{c}} Z_s.$$

We say that  $\underline{\zeta}(K)$  is the *truncated restriction* of  $K$ .

**6.12.** Let  $L \in \mathcal{C}_0^{\mathbf{c}} Z_s$ . We show:

- (a) We have canonically  $\underline{\zeta}(\chi(L)) = \underline{\mathfrak{b}''}(L)$ .

We shall apply the method of [L14, 1.12] with  $\Phi : \mathcal{D}_m(Y_1) \rightarrow \mathcal{D}_m(Y_2)$  replaced by  $\zeta : \mathcal{D}_m(\tilde{G}_s) \rightarrow \mathcal{D}_m(Z_s)$  and with  $\mathcal{D}^{\preceq}(Y_1)$ ,  $\mathcal{D}^{\preceq}(Y_2)$  replaced by  $\mathcal{D}^{\preceq} \tilde{G}_s$ ,  $\mathcal{D}^{\preceq} Z_s$ . We shall take  $\mathbf{X}$  in *loc.cit.* equal to  $\chi(L)$ . The conditions of *loc.cit.* are satisfied: those

concerning  $\mathbf{X}$  are satisfied with  $c' = a + \nu$ , see 6.3. The conditions concerning  $\zeta$  are satisfied with  $c = a + \nu$ , see 6.10. We see that

$$(b) \quad (\zeta(\chi(L)))^j = 0 \text{ if } j > 2a + 2\nu$$

and

$$(c) \quad \underline{gr_{2a+2\nu}((\zeta(\chi(L)))^{2a+2\nu})}(a + \nu) = \underline{\zeta}(\underline{\chi}(L)).$$

Since  $\zeta(\chi(L)) = \mathbf{b}''(L)$ , we see that the left hand side of (c) equals  $\underline{\mathbf{b}}''(L)$ . Thus (a) is proved.

Combining (a) with 4.25(d) and 4.14(d) we see that

$$(b) \text{ we have canonically } \tilde{\epsilon}_s \zeta(\chi(L)) = \underline{\mathbf{h}}(L).$$

**6.13.** Let  $K \in \mathcal{D}(\tilde{G}_s)$  and let  $L \in \mathcal{D}^\bullet \tilde{\mathcal{B}}^2$ . Let  $\tilde{L} = (\mathbf{e}^s)^* L$ . In (a) below the assumption  $s \in \mathbf{Z}_c$  is not used:

(a) *there is a canonical isomorphism  $\tilde{L} \circ \epsilon_s^* \zeta(K) \xrightarrow{\sim} \epsilon_s^* \zeta(K) \circ L$ .*

Let  $Y = \tilde{\mathcal{B}}^2 \times \tilde{G}_s$ . Define  $j : Y \rightarrow \tilde{G}_s$  by  $j(x_0 \mathbf{U}, x_1 \mathbf{U}, \gamma) = \gamma$ . Define  $j_1 : Y \rightarrow \tilde{\mathcal{B}}^2$  by  $j_1(x_0 \mathbf{U}, x_1 \mathbf{U}, \gamma) = (x_0 \mathbf{U}, \gamma^{-1} x_1 \tau^s \mathbf{U})$ . Define  $j_2 : Y \rightarrow \tilde{\mathcal{B}}^2$  by  $j_2(x_0 \mathbf{U}, x_1 \mathbf{U}, \gamma) = (\gamma x_0 \tau^{-s} \mathbf{U}, x_1 \mathbf{U})$ . From the definitions we have  $\tilde{L} \circ \epsilon_s^* \zeta(K) = j_{2!}(j_1^*(\tilde{L}) \otimes j^*(K))$ ,  $\epsilon_s^* \zeta(K) \circ L = j_{2!}(j_2^*(L) \otimes j^*(K))$ . It remains to prove that  $j_1^*(\tilde{L}) = j_2^* L$  that is,  $j_1'^* L = j_2^* L$  where  $j_1' = \mathbf{e}^s j_1 : Y \rightarrow \tilde{\mathcal{B}}^2$  is given by  $j_1'(x_0 \mathbf{U}, x_1 \mathbf{U}, \gamma) = (\tau^s x_0 \tau^{-s} \mathbf{U}, \tau^s \gamma^{-1} x_1 \mathbf{U})$ . The equality  $j_1'^* L = j_2^* L$  follows from the  $G$ -equivariance of  $L$ . This proves (a).

Now let  $K \in \mathcal{C}_0^c \tilde{G}_s$  and let  $L \in \mathcal{C}_0^c \tilde{\mathcal{B}}^2$ . Since  $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$ , we have  $(\mathbf{e}^s)^* L \in \mathcal{C}_0^c \tilde{\mathcal{B}}^2$ , see 3.11(a). We show that

(b) *there is a canonical isomorphism  $(\mathbf{e}^s)^*(L) \circ \tilde{\epsilon}_s \zeta(K) \xrightarrow{\sim} (\tilde{\epsilon}_s \zeta(K)) \circ L$ .*

We apply the method of [L14, 1.12] with  $\Phi : \mathcal{D}_m^\prec \tilde{\mathcal{B}}^2 \rightarrow \mathcal{D}_m^\prec \tilde{\mathcal{B}}^2$ ,  $L' \mapsto L' \circ L$ ,  $\mathbf{X} = \tilde{\epsilon}_s \zeta(K)$  and with  $(c, c') = (a - \nu, \nu + a)$ , see [L16, 2.23(a)], 6.10(c). We deduce that we have canonically

$$(c) \quad \underline{((\tilde{\epsilon}_s \zeta(K))^{\{a+\nu\}} \circ L)^{\{a-\nu\}}} = \underline{(\tilde{\epsilon}_s \zeta(K) \circ L)^{\{2a\}}}.$$

We apply the method of [L14, 1.12] with  $\Phi : \mathcal{D}_m^\prec \tilde{\mathcal{B}}^2 \rightarrow \mathcal{D}_m^\prec \tilde{\mathcal{B}}^2$ ,  $L' \mapsto (\mathbf{e}^s)^* L \circ L'$ ,  $\mathbf{X} = \tilde{\epsilon}_s \zeta(K)$  and with  $(c, c') = (a - \nu, \nu + a)$ , see [L16, 2.23(a)], 6.10(c). We deduce that we have canonically

$$(d) \quad \underline{((\mathbf{e}^s)^* L \circ (\tilde{\epsilon}_s \zeta(K))^{\{a+\nu\}})^{\{a-\nu\}}} = \underline{((\mathbf{e}^s)^* L \circ \tilde{\epsilon}_s \zeta(K))^{\{2a\}}}.$$

We now combine (c),(d) with (a); we obtain (b).

**6.14.** Let  $s', s''$  be integers. Let  $\mu : \tilde{G}_{s'} \times \tilde{G}_{s''} \rightarrow \tilde{G}_{s'+s''}$  be the multiplication map. For  $K \in \mathcal{D}(\tilde{G}_{s'}), K' \in \mathcal{D}(\tilde{G}_{s''})$  (resp.  $K \in \mathcal{D}_m(\tilde{G}_{s'}), K' \in \mathcal{D}_m(\tilde{G}_{s''})$ ) we set  $K * K' = \mu_!(K \boxtimes K')$ ; this is in  $\mathcal{D}(\tilde{G}_{s'+s''})$  (resp. in  $\mathcal{D}_m(\tilde{G}_{s'+s''})$ ). For  $K \in \mathcal{D}(\tilde{G}_{s_1}), K' \in \mathcal{D}(\tilde{G}_{s_2}), K'' \in \mathcal{D}(\tilde{G}_{s_3})$  we have canonically  $(K * K') * K'' = K * (K' * K'')$  (and we denote this by  $K * K' * K''$ ). For  $K \in \mathcal{M}(\tilde{G}_{s'}), K' \in \mathcal{M}(\tilde{G}_{s''})$  we show:

(a) *If  $K'$  is  $G$ -equivariant then we have canonically  $K * K' = ((e^{-s'})^* K') * K'$ . If  $K$  is  $G$ -equivariant then we have canonically  $K * K' = K' * ((e^{s''})^* K)$ .*  
The proof is immediate. It will be omitted. (Compare [L14, 4.1].)

**6.15.** Let  $s', s'' \in \mathbf{Z}$ . We show:

(a) *For  $K \in \mathcal{D}(\tilde{G}_{s'}), L \in \mathcal{D}(Z_{s''})$  we have canonically  $K * \chi(L) = \chi(L \bullet \zeta(K))$ .*  
Let  $Y = \tilde{G}_{s'} \times \tilde{G}_{s''} \times \mathcal{B}$ . Define  $c : Y \rightarrow \tilde{G}_{s'} \times Z_{s''}$  by

$$c(\gamma_1, \gamma_2, B) = (\gamma_1, (B, \gamma_2 B \gamma_2^{-1}, \gamma_2 U_B));$$

define  $d : Y \rightarrow \tilde{G}_{s'+s''}$  by  $d(\gamma_1, \gamma_2, B) = \gamma_1 \gamma_2$ . From the definitions we see that both  $K * \chi(L), \chi(L \bullet \zeta(K))$  can be identified with  $d_! c^*(K \boxtimes L)$ . This proves (a).

Now let  $L \in \mathcal{D}(Z_{s'}), L' \in \mathcal{D}(Z_{s''})$ . Replacing in (a)  $K, L$  by  $\chi(L), L'$  and using 6.8(a), we obtain

$$(b) \quad \chi(L) * \chi(L') = \chi(L' \bullet \mathbf{b}''(L)).$$

**6.16.** Let  $s' \in \mathbf{Z}_c$ . Let  $L \in \mathcal{D}^\spadesuit(Z_s), L' \in \mathcal{D}^\spadesuit(Z_{s'}), j \in \mathbf{Z}$ . We show:

(a) *If  $L \in \mathcal{D}^\preceq Z_s$  or  $L' \in \mathcal{D}^\preceq Z_{s'}$  then  $L' \bullet \mathbf{b}''(L) \in \mathcal{D}^\preceq Z_{s+s'}$ .*  
(b) *If  $L \in \mathcal{D}^\prec Z_s$  or  $L' \in \mathcal{D}^\prec Z_{s'}$  then  $L' \bullet \mathbf{b}''(L) \in \mathcal{D}^\prec Z_{s+s'}$ .*  
(c) *If  $L \in \mathcal{M}^\preceq Z_s, L' \in \mathcal{M}^\spadesuit Z_{s'}$  and  $j > 3a + \rho + \nu$  then  $(L' \bullet \mathbf{b}''(L))^j \in \mathcal{D}^\prec Z_{s+s'}$ .*  
Now (a),(b) follow from 4.25(b) and 4.23(a). To prove (c) we may assume that  $L = \mathbb{L}_{\lambda,s}^{\dot{w}}, L' = \mathbb{L}_{\lambda',s'}^{\dot{w}'}$  with  $w \cdot \lambda \in I_n^s, w' \cdot \lambda' \in I_n^{s'}$  and  $w \cdot \lambda \preceq \mathbf{c}$ . We apply the method of [L14, 1.12] with  $\Phi : \mathcal{D}^\preceq Z_s \rightarrow \mathcal{D}^\preceq Z_{s+s'}, L_1 \mapsto L' \bullet L_1$  and  $\mathbf{X} = \mathbf{b}''(L)$  and with  $c' = 2\nu + 2a$  (see 4.25(c)),  $c = a + \rho - \nu$  (see 4.23(b)). We have  $c + c' = \nu + \rho + 3a$  hence (c) holds.

**6.17.** Let  $s' \in \mathbf{Z}_c$ . Let  $L \in \mathcal{D}^\spadesuit(Z_s), L' \in \mathcal{D}^\spadesuit(Z_{s'}), j \in \mathbf{Z}$ . We show:

(a) *If  $L \in \mathcal{D}^\preceq Z_s$  or  $L' \in \mathcal{D}^\preceq Z_{s'}$  then  $\chi(L' \bullet \mathbf{b}''(L)) \in \mathcal{D}^\preceq \tilde{G}_{s+s'}$ .*  
(b) *If  $L \in \mathcal{D}^\prec Z_s$  or  $L' \in \mathcal{D}^\prec Z_{s'}$  then  $\chi(L' \bullet \mathbf{b}''(L)) \in \mathcal{D}^\prec \tilde{G}_{s+s'}$ .*  
(c) *If  $L \in \mathcal{M}^\preceq Z_s, L' \in \mathcal{M}^\spadesuit Z_{s'}$  and  $j > 4a + 2\nu + \rho$  then  $(\chi(L' \bullet \mathbf{b}''(L)))^j \in \mathcal{M}^\prec \tilde{G}_{s+s'}$ .*

(a),(b) follow from 6.3(a) using 6.16(a),(b). To prove (c) we can assume that  $L = \mathbb{L}_{\lambda,s}^{\dot{w}}, L' = \mathbb{L}_{\lambda',s'}^{\dot{w}'}$  with  $w \cdot \lambda \in I_n^s, w' \cdot \lambda' \in I_n^{s'}$  and  $w \cdot \lambda \preceq \mathbf{c}$ . We apply the method of [L14, 1.12] with  $\Phi : \mathcal{D}^\preceq Z_{s+s'} \rightarrow \mathcal{D}^\preceq \tilde{G}_{s+s'}, L_1 \mapsto \chi(L_1), \mathbf{X} = L' \bullet \mathbf{b}''(L)$  and with  $c' = \nu + \rho + 3a$  (see 6.16(c)),  $c = a + \nu$  (see 6.3(b)). We have  $c + c' = 2\nu + \rho + 4a$  hence (c) holds.

**6.18.** Let  $s' \in \mathbf{Z}_{\mathbf{c}}$ . Let  $K \in \mathcal{D}^\bullet(\tilde{G}_s)$ ,  $K' \in \mathcal{D}^\bullet(\tilde{G}_{s'})$ . We show:

- (a) If  $K \in \mathcal{D}^\preceq \tilde{G}_s$  or  $K' \in \mathcal{D}^\preceq \tilde{G}_{s'}$ , then  $K * K' \in \mathcal{D}^\preceq \tilde{G}_{s+s'}$ .
- (b) If  $K \in \mathcal{D}^\prec \tilde{G}_s$  or  $K' \in \mathcal{D}^\prec \tilde{G}_{s'}$ , then  $K * K' \in \mathcal{D}^\prec \tilde{G}_{s+s'}$ .
- (c) If  $K \in \mathcal{D}^\preceq \tilde{G}_s$  or  $K' \in \mathcal{D}^\preceq \tilde{G}_{s'}$  and  $j > 2a + \rho$  then  $(K * K')^j \in \mathcal{D}^\prec \tilde{G}_{s+s'}$ .

We can assume that  $K = A \in CS_{\mathbf{o},s}$ ,  $K' = A' \in CS_{\mathbf{o},s'}$ . Let  $A'' \in \mathcal{M}(\tilde{G}_{s+s'})$  be a composition factor of  $(A * A')^j$ . By 6.2(c) we can find  $w \cdot \lambda \in \mathbf{c}_A$ ,  $w' \cdot \lambda' \in \mathbf{c}_{A'}$  such that  $(A : (R_{\lambda,s}^{\dot{w}})^{n_w}) \neq 0$ ,  $(A' : (R_{\lambda',s'}^{\dot{w}'})^{n_{w'}}) \neq 0$ . Then  $A$  is a direct summand of  $R_{\lambda,s}^{\dot{w}}[n_w]$  and  $A'$  is a direct summand of  $R_{\lambda',s'}^{\dot{w}'}[n_{w'}]$ . Hence  $A * A'$  is a direct summand of

$$R_{\lambda,s}^{\dot{w}} * R_{\lambda',s'}^{\dot{w}'}[a(w \cdot \lambda) + a(w' \cdot \lambda') + |w| + |w'| + 2\Delta]$$

and  $(A * A')^j$  is a direct summand of

$$\begin{aligned} & (R_{\lambda,s}^{\dot{w}} * R_{\lambda',s'}^{\dot{w}'}[|w| + |w'| + 2\nu + 2\rho])^{j+a(w \cdot \lambda)+a(w' \cdot \lambda')+2\nu} \\ &= (\chi(\mathbb{L}_{\lambda,s}^{\dot{w}}) * \chi(\mathbb{L}_{\lambda',s'}^{\dot{w}'}))^{j+a(w \cdot \lambda)+a(w' \cdot \lambda')+2\nu}. \end{aligned}$$

Using 6.15(b) we see that  $(A * A')^j$  is a direct summand of

$$(d) \quad (\chi(\mathbb{L}_{\lambda',s'}^{\dot{w}'} \bullet \mathbf{b}''(\mathbb{L}_{\lambda,s}^{\dot{w}})))^{j+a(w \cdot \lambda)+a(w' \cdot \lambda')+2\nu}.$$

Hence  $A''$  is a composition factor of (d). Using 6.17(a) we see that  $A'' \in CS_{\mathbf{o},s+s'}$ , that  $\mathbf{c}_{A''} \preceq w \cdot \lambda$  and that  $\mathbf{c}_{A''} \preceq w' \cdot \lambda'$ . In the setup of (a) we have  $w \cdot \lambda \preceq \mathbf{c}$  or  $w' \cdot \lambda' \preceq \mathbf{c}$  hence  $\mathbf{c}_{A''} \leq \mathbf{c}$ . Thus (a) holds. Similarly, (b) holds. In the setup of (c) we have  $w \cdot \lambda \preceq \mathbf{c}$  and  $w' \cdot \lambda' \preceq \mathbf{c}$ . Hence  $a(w \cdot \lambda) \geq a$ ,  $a(w' \cdot \lambda') \geq a$ . (See Q3 in 1.9.) Assume that  $\mathbf{c}_{A''} = \mathbf{c}$ . Since  $A''$  is a composition factor of (d), we see from 6.17(c) that

$$j + a(w \cdot \lambda) + a(w' \cdot \lambda') + 2\nu \leq 4a + 2\nu + \rho$$

hence  $j + 2a + 2\nu \leq 4a + 2\nu + \rho$  and  $j \leq 2a + \rho$ . This proves (c).

**6.19.** Let  $s' \in \mathbf{Z}_{\mathbf{c}}$ . For  $K \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_s$ ,  $K' \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_{s'}$ , we set

$$K \underline{*} K' = \underline{(K * K')^{\{2a+\rho\}}} \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_{s+s'}.$$

We say that  $K \underline{*} K'$  is the *truncated convolution* of  $K, K'$ . Note that 6.14(a) induces for  $K, K' \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}$  a canonical isomorphism

$$(a) \quad K \underline{*} K' = K' \underline{*} ((\mathbf{e}^{s'})^* K).$$

Let  $L \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_{s'}$ ,  $K \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_s$ . Using the method of [L14, 1.12] several times, we see that

$$K \underline{*} \chi(L) = \underline{gr_k((K * \chi(L))^k)}(k/2)$$



where  $k = (a + \nu) + (2a + \rho) = 3a + \nu + \rho$  and

$$\underline{\chi}(L \bullet \underline{\zeta}(K)) = \underline{gr}_{k'}((\underline{\chi}(L \bullet \zeta(K)))^{k'})(k'/2)$$

where  $k' = (a + \nu) + (a + \nu) + (a + \rho - \nu) = 3a + \nu + \rho$ . Using now 6.15(a) and the equality  $k = k'$  we obtain

$$(b) \quad K \underline{*} \underline{\chi}(L) = \underline{\chi}(L \bullet \underline{\zeta}(K)).$$

Let  $L \in \mathcal{C}_0^c Z_s$ ,  $L' \in \mathcal{C}_0^c Z_{s'}$ . Using the method of [L14, 1.12] several times, we see that

$$\underline{\chi}(L) \underline{*} \underline{\chi}(L') = \underline{gr}_k((\underline{\chi}(L) * \underline{\chi}(L'))^k)(k/2)$$

where  $k = (a + \nu) + (a + \nu) + (2a + \rho) = 4a + 2\nu + \rho$  and

$$\underline{\chi}(L' \bullet \underline{\mathfrak{b}}''(L)) = \underline{gr}_{k'}((\underline{\chi}(L' \bullet \mathfrak{b}''(L)))^{k'})(k'/2)$$

where  $k' = (2a + 2\nu) + (a + \rho - \nu) + (a + \nu) = 4a + 2\nu + \rho$ . Using now 6.15(b) and the equality  $k = k'$  we obtain

$$(c) \quad \underline{\chi}(L) \underline{*} \underline{\chi}(L') = \underline{\chi}(L' \bullet \underline{\mathfrak{b}}''(L)).$$

We show (assuming that  $s_h \in \mathbf{Z}_c$  for  $h = 1, 2, 3$ ):

(d) For  $K \in \mathcal{C}_0^c \tilde{G}_{s_1}$ ,  $K' \in \mathcal{C}_0^c \tilde{G}_{s_2}$ ,  $K'' \in \mathcal{C}_0^c \tilde{G}_{s_3}$ , there is a canonical isomorphism  $(K \underline{*} K') \underline{*} K'' \xrightarrow{\sim} K \underline{*} (K' \underline{*} K'')$ .

Indeed, just as in [L14, 4.7] we can identify, using the method of [L14, 1.12], both  $(K \underline{*} K') \underline{*} K''$  and  $K \underline{*} (K' \underline{*} K'')$  with  $\underline{(K * K' * K'')^{\{4a+2\rho\}}}$ .

**6.20.** Let  $s', s'' \in \mathbf{Z}$ . For  $K \in \mathcal{D}(\tilde{G}_{s'})$ ,  $K' \in \mathcal{D}(\tilde{G}_{s''})$ , we show:

(a) We have canonically  $\zeta(K * K') = \zeta(K') \bullet \zeta(K)$ .

Let

$$Y = \{(B, \gamma U_B, \gamma_1, \gamma_2); B \in \mathcal{B}, \gamma \in \tilde{G}_{s'+s''}, \gamma_1 \in \tilde{G}_{s'}, \gamma_2 \in \tilde{G}_{s''}; \gamma_1 \gamma_2 \in \gamma U_B\}.$$

Define  $j_1 : Y \rightarrow \tilde{G}_{s'}$ ,  $j_2 : Y \rightarrow \tilde{G}_{s''}$  by  $j_1(B, \gamma U_B, \gamma_1, \gamma_2) = \gamma_1$ ,  $j_2(B, \gamma U_B, \gamma_1, \gamma_2) = \gamma_2$ . Define  $j : Y \rightarrow Z_{s'+s''}$  by  $j(B, \gamma U_B, \gamma_1, \gamma_2) = (B, \gamma B \gamma^{-1}, \gamma U_B)$ . From the definitions we have  $\zeta(K * K') = j_!(j_1^*(K) \otimes j_2^*(K')) = \zeta(K') \bullet \zeta(K)$ ; (a) follows. ■

Let  $s' \in \mathbf{Z}_c$ . For  $K \in \mathcal{D}_0^c(G_s)$ ,  $K' \in \mathcal{D}_0^c(G_{s'})$ , we show:

(b) We have canonically  $\underline{\zeta}(K \underline{*} K') = \underline{\zeta}(K') \bullet \underline{\zeta}(K)$ .

Using the method of [L14, 1.12] we see that

$$\underline{\zeta}(K \underline{*} K') = \underline{gr}_k((\underline{\zeta}(K * K'))^k)(k/2)$$

where  $k = (a + \nu) + (2a + \rho) = 3a + \nu + \rho$  and that

$$\underline{\zeta}(K') \bullet \underline{\zeta}(K) = \underline{gr}_{k'}((\underline{\zeta}(K) \bullet \underline{\zeta}(K'))^{k'})(k'/2)$$

where  $k' = (a + \rho - \nu) + (a + \nu) + (a + \nu) = 3a + \nu + \rho$ . It remains to use (a) and the equality  $k = k'$ .

**6.21.** Let  $s' \in \mathbf{Z}$ . Define  $h : \tilde{G}_{s'} \rightarrow \tilde{G}_{-s'}$  by  $\gamma \mapsto \gamma^{-1}$ . For  $K \in \mathcal{D}(\tilde{G}_{-s'})$  we set  $K^\dagger = h^*K \in \mathcal{D}(\tilde{G}_{s'})$ . We show:

(a) For  $L \in \mathcal{D}(Z_{-s'})$  we have  $(\chi(L))^\dagger = \chi(L^\dagger)$  with  $L^\dagger$  as in 4.2.

This follows from the definition of  $\chi$  using the commutative diagram

$$\begin{array}{ccccc} Z_{s'} & \xleftarrow{f} & \dot{Z}_{s'} & \xrightarrow{\pi} & \tilde{G}_{s'} \\ \mathfrak{h} \downarrow & & \dot{\mathfrak{h}} \downarrow & & h \downarrow \\ Z_{-s'} & \xleftarrow{f} & \dot{Z}_{-s'} & \xrightarrow{\pi} & \tilde{G}_{-s'} \end{array}$$

where  $f, \pi$  are as in 6.1,  $\mathfrak{h}$  is as in 4.2 and  $\dot{\mathfrak{h}} : \dot{Z}_{s'} \rightarrow \dot{Z}_{-s'}$  is  $(B, B', \gamma) \mapsto (B', B, \gamma^{-1})$ .

From (a) and 4.3(e) we see that, if  $w \cdot \lambda \in I_n^{-s}$ , then

$$(b) \quad (\chi(\mathbb{L}_{\lambda, -s}^{\dot{w}}))^\dagger = \chi(\mathbb{L}_{w(\lambda)^{-1}, s}^{\dot{w}^{-1}}).$$

We deduce that

(c) if  $A \in CS_{\mathbf{c}, -s}$ , then  $A^\dagger \in CS_{\tilde{\mathbf{c}}, s}$ .

From (a), (c) we deduce:

(d) For  $L \in \mathcal{C}_0^{\mathbf{c}} Z_{-s}$  we have  $(\underline{\chi}(L))^\dagger = \underline{\chi}(L^\dagger)$  where the second  $\underline{\chi}$  is relative to  $\tilde{\mathbf{c}}, \mathfrak{o}^{-1}$  instead of  $\mathbf{c}, \mathfrak{o}$ .

## 7. EQUIVALENCE OF $\mathcal{C}^{\mathbf{c}} \tilde{G}_s$ WITH THE $\mathbf{e}^s$ -CENTRE OF $\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2$

**7.1.** In this section (except in 7.8) let  $\mathbf{c}, \mathfrak{o}, a, n, \Psi$  be as in 3.1(a).

In this subsection we assume that  $s \in \mathbf{Z}_{\mathbf{c}}$ . Let  $u : \tilde{G}_{-s} \rightarrow \mathbf{p}$  be the obvious map; let  $\phi : \mathbf{p} \rightarrow G$  be the map with image  $\{1\}$ . From [L5, 7.4] we see that for  $K, K'$  in  $\mathcal{M}_m \tilde{G}_{-s}$  we have canonically

$$(u_!(K \otimes K'))^0 = \text{Hom}_{\mathcal{M}(\tilde{G}_{-s})}(\mathfrak{D}(K), K'), \quad (u_!(K \otimes K'))^j = 0 \text{ if } j > 0.$$

We deduce that if  $K, K'$  are also pure of weight 0 then  $(u_!(K \otimes K'))^0$  is pure of weight 0 that is,  $(u_!(K \otimes K'))^0 = gr_0(u_!(K \otimes K'))^0$ . From the definitions we see that we have  $u_!(K \otimes K') = \phi^*(K^\dagger * K')$  where  $K^\dagger \in \mathcal{M}_m(\tilde{G}_s)$  is as in 6.21. Hence, for  $K'$  in  $\mathcal{C}_0^{\mathbf{c}} \tilde{G}_{-s}$  and  $K$  in  $\mathcal{C}_0^{\tilde{\mathbf{c}}} \tilde{G}_{-s}$  (so that  $K^\dagger \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_s$ , see 6.21(c)) we have

$$(a) \quad \text{Hom}_{\mathcal{M}(\tilde{G}_{-s})}(\mathfrak{D}(K), K') = (\phi^*(K^\dagger * K'))^0 = (\phi^*(K^\dagger * K'))^{\{0\}}.$$

Using [L14, 8.2] with  $\Phi : \mathcal{D}_m^{\leq} \tilde{G}_0 \rightarrow \mathcal{D}_m \mathbf{p}$ ,  $K_1 \mapsto \phi^* K_1$ ,  $c = -2a - \rho$  (see [L16, 6.8(a)]),  $K$  replaced by  $K^\dagger * K' \in \mathcal{D}_m(\tilde{G}_0)$  and  $c' = 2a + \rho$ , we see that we have canonically

$$(\phi^*(K^\dagger * K'))^{\{-2a-\rho\}} \subset (\phi^*(K^\dagger * K'))^{\{0\}}.$$

In particular, if  $L \in \mathcal{C}_0^c Z_{-s}$ ,  $L' \in \mathcal{C}_0^c Z_s$ , then we have canonically

$$(\phi^*(\underline{\chi}(L') * \underline{\chi}(L)))^{\{-2a-\rho\}} \subset (\phi^*(\underline{\chi}(L') * \underline{\chi}(L)))^{\{0\}}.$$

Using the equality

$$(\phi^*(\underline{\chi}(L') * \underline{\chi}(L)))^{\{-2a-\rho\}} = \phi^*(\underline{\chi}(L \bullet \underline{\zeta}(\underline{\chi}(L'))))^{\{-2a-\rho\}}$$

which comes from 6.19(b), we deduce that we have canonically

$$\phi^*(\underline{\chi}(L \bullet \underline{\zeta}(\underline{\chi}(L'))))^{\{-2a-\rho\}} \subset (\phi^*(\underline{\chi}(L') * \underline{\chi}(L)))^{\{0\}},$$

or equivalently, using (a) with  $K, K'$  replaced by  $\underline{\chi}(L')^\dagger, \underline{\chi}(L)$ ,

$$\begin{aligned} \phi^*(\underline{\chi}(L \bullet \underline{\zeta}(\underline{\chi}(L'))))^{\{-2a-\rho\}} &\subset \text{Hom}_{\mathcal{C}^c \tilde{G}_{-s}}(\mathfrak{D}(\underline{\chi}(L')^\dagger), \underline{\chi}(L)) \\ &= \text{Hom}_{\mathcal{C}^c \tilde{G}_s}(\mathfrak{D}(\underline{\chi}(L)^\dagger), \underline{\chi}(L')). \end{aligned}$$

Using now [L16, 6.9(d)] with  $L$  replaced by  $L \bullet \underline{\zeta}(\underline{\chi}(L')) \in \mathcal{C}_0^c Z_0$ , we have canonically

$$\phi^*(\underline{\chi}(L \bullet \underline{\zeta}(\underline{\chi}(L'))))^{\{-2a-\rho\}} = \text{Hom}_{\mathcal{C}^c Z_0}(\mathbf{1}'_0, L \bullet \underline{\zeta}(\underline{\chi}(L'))).$$

Thus we have canonically

$$\text{Hom}_{\mathcal{C}^c Z_0}(\mathbf{1}'_0, L \bullet \underline{\zeta}(\underline{\chi}(L'))) \subset \text{Hom}_{\mathcal{C}^c \tilde{G}_s}(\mathfrak{D}(\underline{\chi}(L)^\dagger), \underline{\chi}(L'))$$

or equivalently (using 5.8(a))

$$\text{Hom}_{\mathcal{C}^c Z_{-s}}(\mathfrak{D}(\underline{\zeta}(\underline{\chi}(L'))^\dagger), L) \subset \text{Hom}_{\mathcal{C}^c \tilde{G}_s}(\mathfrak{D}(\underline{\chi}(L)^\dagger), \underline{\chi}(L')).$$

Now we have

$$\begin{aligned} \text{Hom}_{\mathcal{C}^c Z_{-s}}(\mathfrak{D}(\underline{\zeta}(\underline{\chi}(L'))^\dagger), L) &= \text{Hom}_{\mathcal{C}^c \tilde{Z}_{-s}}(\mathfrak{D}(L), \underline{\zeta}(\underline{\chi}(L'))^\dagger) \\ &= \text{Hom}_{\mathcal{C}^c Z_s}((\mathfrak{D}(L))^\dagger, \underline{\zeta}(\underline{\chi}(L'))), \end{aligned}$$

hence

$$\text{Hom}_{\mathcal{C}^c Z_s}((\mathfrak{D}(L))^\dagger, \underline{\zeta}(\underline{\chi}(L'))) \subset \text{Hom}_{\mathcal{C}^c \tilde{G}_s}(\mathfrak{D}(\underline{\chi}(L)^\dagger), \underline{\chi}(L')).$$

We set  ${}^1L = \mathfrak{D}(L)^\dagger = (\mathfrak{D}(L))^\dagger \in \mathcal{C}_0^c Z_s$  and note that

$$\mathfrak{D}(\underline{\chi}(L)^\dagger) = \mathfrak{D}(\underline{\chi}(L^\dagger)) = \underline{\chi}(\mathfrak{D}(L^\dagger)) = \underline{\chi}({}^1L),$$

see 6.21(d), 6.7(b). We obtain

$$(b) \quad \text{Hom}_{\mathcal{C}^c Z_s}({}^1L, \underline{\zeta}(\underline{\chi}(L'))) \subset \text{Hom}_{\mathcal{C}^c \tilde{G}_s}(\underline{\chi}({}^1L), \underline{\chi}(L'))$$

for any  ${}^1L, L'$  in  $\mathcal{C}_0^{\mathbf{c}}Z_s$ . We show that (b) is an equality:

$$(c) \quad \text{Hom}_{\mathcal{C}^{\mathbf{c}}Z_s}({}^1L, \zeta(\underline{\chi}(L'))) = \text{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}({}^1L), \underline{\chi}(L')).$$

Let  $N'$  (resp.  $N''$ ) be the dimension of the left (resp. right) hand side of (b). It is enough to show that  $N' = N''$ . We can assume that  ${}^1L = \mathbb{L}_{\lambda', s}^{\dot{z}'}$ ,  $L' = \mathbb{L}_{\lambda, s}^{\dot{z}}$  where  $z \cdot \lambda \in \mathbf{c}^s$ ,  $z'' \cdot \lambda' \in \mathbf{c}^s$ . By 6.12(a),  $N'$  is the multiplicity of  ${}^1L$  in  $\underline{\mathbf{b}}''(L')$ ; by the fully faithfulness of  $\tilde{\epsilon}_s$  this is the same as the multiplicity of  $\tilde{\epsilon}_s {}^1L$  in  $\tilde{\epsilon}_s \underline{\mathbf{b}}''(L') = \underline{\mathbf{b}}'(L') = \underline{\mathbf{b}}(L')$  (the last two equalities use 4.25(d) and 4.14(d)). By 4.13(d), this is the same as the multiplicity of  $\mathbf{L}_{\lambda'}^{\dot{z}'}$  in

$$\bigoplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \circ \mathbf{L}_{\lambda}^{\dot{z}} \circ \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}.$$

Using now [L16, 2.22(c)] we see that  $N'$  is the coefficient of  $t_{z' \cdot \lambda'}$  in

$$\sum_{y \in W; y \cdot \lambda \in \mathbf{c}} t_{\mathbf{e}^{-s}(y) \cdot \mathbf{e}^{-s}(\lambda)} t_{z \cdot \lambda} t_{y^{-1} \cdot y(\lambda)} \in \mathbf{H}^{\infty}.$$

Hence if  $\mathbf{t} : \mathbf{H}^{\infty} \rightarrow \mathbf{Z}$  is as in 1.9, then

$$N' = \sum_{y \in W; y \cdot \lambda \in \mathbf{c}} \mathbf{t}(t_{\mathbf{e}^{-s}(y) \cdot \mathbf{e}^{-s}(\lambda)} t_{z \cdot \lambda} t_{y^{-1} \cdot y(\lambda)} t_{z'^{-1} \cdot z'(\lambda')}).$$

This can be rewritten as

$$N' = \sum_{y \cdot \lambda_1 \in \mathbf{c}} \mathbf{t}(t_{\mathbf{e}^{-s}(y) \cdot \mathbf{e}^{-s}(\lambda_1)} t_{z \cdot \lambda} t_{y^{-1} \cdot y(\lambda_1)} t_{z'^{-1} \cdot z'(\lambda')}).$$

(In the last sum, the terms corresponding to  $y \cdot \lambda_1$  with  $\lambda_1 \neq \lambda$  are equal to zero.) By 6.6(c) (with  $z \cdot \lambda, z' \cdot \lambda'$  interchanged) we have

$$N'' = \sum_{y \cdot \lambda_1 \in \mathbf{c}} \mathbf{t}(t_{\mathbf{e}^{-s}(y) \cdot \mathbf{e}^{-s}(\lambda_1)} t_{z \cdot \lambda} t_{y^{-1} \cdot y(\lambda_1)} t_{z'^{-1} \cdot z'(\lambda)}).$$

Thus,  $N' = N''$ . This completes the proof of (c).

**7.2.** Let  $s, s' \in \mathbf{Z}_{\mathbf{c}}$ . We define a bifunctor  $\mathcal{C}^{\mathbf{c}}\tilde{G}_s \times \mathcal{C}^{\mathbf{c}}\tilde{G}_{s'} \rightarrow \mathcal{C}^{\mathbf{c}}\tilde{G}_{s+s'}$  denoted by  $K, K' \mapsto K \underline{*} K'$  as follows. By replacing if necessary  $\Psi$  in 7.1 by a power, we can assume that any  $A \in CS_{\mathbf{c}, s}$  and any  $A \in CS_{\mathbf{c}, s'}$  admits a mixed structure (defined in terms of  $\Psi$ ) of pure weight zero. Let  $K \in \mathcal{C}^{\mathbf{c}}\tilde{G}_s$ ,  $K' \in \mathcal{C}^{\mathbf{c}}\tilde{G}_{s'}$ ; we choose mixed structures of pure weight 0 on  $K, K'$  with respect to  $\Psi$  (this is possible by our choice of  $\Psi$ ). We define  $K \underline{*} K'$  as in 6.19 in terms of these mixed structures and we then disregard the mixed structure on  $K \underline{*} K'$ . The resulting object of  $\mathcal{C}^{\mathbf{c}}\tilde{G}_{s+s'}$  is denoted again by  $K \underline{*} K'$ ; it is independent of the choice made.

In the same way the functor  $\underline{\chi} : \mathcal{C}_0^{\mathbf{c}} Z_s \rightarrow \mathcal{C}_0^{\mathbf{c}} \tilde{G}_s$  gives rise to a functor  $\mathcal{C}^{\mathbf{c}} Z_s \rightarrow \mathcal{C}^{\mathbf{c}} \tilde{G}_s$  denoted again by  $\underline{\chi}$ ; the functor  $\underline{\zeta} : \mathcal{C}_0^{\mathbf{c}} \tilde{G}_s \rightarrow \mathcal{C}_0^{\mathbf{c}} Z_s$  gives rise to a functor  $\mathcal{C}^{\mathbf{c}} \tilde{G}_s \rightarrow \mathcal{C}^{\mathbf{c}} Z_s$  denoted again by  $\underline{\zeta}$ .

The operation  $K * K'$  is again called truncated convolution. It has a canonical associativity isomorphism (deduced from that in 6.19(d)); this makes  $\sqcup_{s \in \mathbf{Z}_c} \mathcal{C}^{\mathbf{c}} \tilde{G}_s$  into a monoidal category.

From 6.20 we see that under  $\underline{\zeta} : \sqcup_{s \in \mathbf{Z}_c} \mathcal{C}^{\mathbf{c}} \tilde{G}_s \rightarrow \sqcup_{s \in \mathbf{Z}_c} \mathcal{C}^{\mathbf{c}} Z_s$ , the monoidal structure on  $\sqcup_{s \in \mathbf{Z}_c} \mathcal{C}^{\mathbf{c}} \tilde{G}_s$  is compatible with the opposite of the monoidal structure on  $\sqcup_{s \in \mathbf{Z}_c} \mathcal{C}^{\mathbf{c}} Z_s$ .

If  $K \in \mathcal{C}^{\mathbf{c}} \tilde{G}_s$  then the isomorphisms 6.13(b) provide an  $\mathbf{e}^s$ -half-braiding for  $\tilde{\epsilon}_s \underline{\zeta}(K) \in \mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2$  so that  $\tilde{\epsilon}_s \underline{\zeta}(K)$  can be naturally viewed as an object of  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$  denoted by  $\overline{\tilde{\epsilon}_s \underline{\zeta}(K)}$ . (Note that 6.13(b) is stated in the mixed category but it implies the corresponding result in the unmixed category.) Then  $K \mapsto \overline{\tilde{\epsilon}_s \underline{\zeta}(K)}$  is a functor  $\mathcal{C}^{\mathbf{c}} \tilde{G}_s \rightarrow \mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$ .

**Theorem 7.3.** *Let  $s \in \mathbf{Z}_c$ . The functor  $\mathcal{C}^{\mathbf{c}} \tilde{G}_s \rightarrow \mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$ ,  $K \mapsto \overline{\tilde{\epsilon}_s \underline{\zeta}(K)}$  is an equivalence of categories.*

From 6.12(a), 4.14(d), 4.25(d) we have canonically for any  $z \cdot \lambda \in \mathbf{c}^s$ :

$$(a) \quad \tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}})) = \underline{\mathbf{h}}(\mathbb{L}_{\lambda,s}^{\dot{z}})$$

as objects of  $\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2$ . From the definitions we see that the  $\mathbf{e}^s$ -half-braiding on the left hand side of (a) provided by 7.2 is the same as the  $\mathbf{e}^s$ -half-braiding on the right hand side of (a) provided by 4.14(j). Hence we have

$$(b) \quad \overline{\tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}))} = \overline{\underline{\mathbf{h}}(\mathbb{L}_{\lambda,s}^{\dot{z}})}$$

as objects of  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$ . Using this and 5.7(a) with  $L' = \tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{w}}))$  (where  $z \cdot \lambda, w \cdot \lambda'$  are in  $\mathbf{c}^s$ ), we have

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2}(\mathbb{L}_{\lambda}^{\dot{z}}, \tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{w}}))) = \mathrm{Hom}_{\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}}(\overline{\tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}))}, \overline{\tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{w}}))}).$$

Combining this with the equalities

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{w}})) &= \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} Z_s}(\mathbb{L}_{l,s}^{\dot{z}}, \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{w}}))) \\ &= \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2}(\mathbb{L}_l^{\dot{z}}, \tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{w}}))), \end{aligned}$$

of which the first comes from 6.10(c) and the second comes from the fully faithfulness of  $\tilde{\epsilon}_s$ , we obtain

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{w}})) = \mathrm{Hom}_{\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}}(\overline{\tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}))}, \overline{\tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(\mathbb{L}_{\lambda',s}^{\dot{w}}))}).$$

In other words, setting

$$\mathbf{A}_{z \cdot \lambda, w \cdot \lambda'} = \text{Hom}_{\mathcal{C}^c \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}}), \underline{\chi}(\mathbb{L}_{\lambda', s}^{\dot{w}})),$$

$$\mathbf{A}'_{z \cdot \lambda, w \cdot \lambda'} = \text{Hom}_{\mathcal{Z}_{\mathbf{e}^s}^c}(\overline{\tilde{\epsilon}_s \zeta(\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}}))}, \overline{\tilde{\epsilon}_s \zeta(\underline{\chi}(\mathbb{L}_{\lambda', s}^{\dot{w}}))}),$$

we have

$$(c) \quad \mathbf{A}_{z \cdot \lambda, w \cdot \lambda'} = \mathbf{A}'_{z \cdot \lambda, w \cdot \lambda'}.$$

Note that the identification (c) is induced by the functor  $K \mapsto \overline{\tilde{\epsilon}_s \zeta(K)}$ . Let  $\mathbf{A} = \oplus \mathbf{A}_{z \cdot \lambda, w \cdot \lambda'}$ ,  $\mathbf{A}' = \oplus \mathbf{A}'_{z \cdot \lambda, w \cdot \lambda'}$  (both direct sums are taken over all  $z \cdot \lambda, w \cdot \lambda'$  in  $\mathbf{c}^s$ ). Then from (c) we have  $\mathbf{A} = \mathbf{A}'$ . Note that this identification is compatible with the obvious algebra structures of  $\mathbf{A}, \mathbf{A}'$ .

For any  $A \in CS_{\mathbf{c}, s}$  we denote by  $\mathbf{A}_A$  the set of all  $f \in \mathbf{A}$  such that for any  $z \cdot \lambda, w \cdot \lambda'$ , the  $(z \cdot \lambda, w \cdot \lambda')$ -component of  $f$  maps the  $A$ -isotypic component of  $\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}})$  to the  $A$ -isotypic component of  $\underline{\chi}(\mathbb{L}_{\lambda', s}^{\dot{w}})$  and any other isotypic component of  $\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}})$  to 0. Thus,  $\mathbf{A} = \oplus_{A \in CS_{\mathbf{c}, s}} \mathbf{A}_A$  is the decomposition of  $\mathbf{A}$  into a sum of simple algebras. (Each  $\mathbf{A}_A$  is nonzero since, by 6.2(c) and 6.5(a), any  $A$  is a summand of some  $\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}})$ .)

Let  $\mathfrak{S}$  be a set of representatives for the isomorphism classes of simple objects of  $\mathcal{Z}_{\mathbf{e}^s}^c$ . For any  $\sigma \in \mathfrak{S}$  we denote by  $\mathbf{A}'_{\sigma}$  the set of all  $f' \in \mathbf{A}'$  such that for any  $z \cdot \lambda, w \cdot \lambda'$ , the  $(z \cdot \lambda, w \cdot \lambda')$ -component of  $f'$  maps the  $\sigma$ -isotypic component of  $\overline{\tilde{\epsilon}_s \zeta(\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}}))}$  to the  $\sigma$ -isotypic component of  $\overline{\tilde{\epsilon}_s \zeta(\underline{\chi}(\mathbb{L}_{\lambda', s}^{\dot{w}}))}$  and all other isotypic components of  $\overline{\tilde{\epsilon}_s \zeta(\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}}))}$  to zero. Then  $\mathbf{A}' = \oplus_{\sigma \in \mathfrak{S}} \mathbf{A}'_{\sigma}$  is the decomposition of  $\mathbf{A}'$  into a sum of simple algebras. (Each  $\mathbf{A}'_{\sigma}$  is nonzero since any  $\sigma$  is a summand of some  $\overline{\tilde{\epsilon}_s \zeta(\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}}))}$  with  $z \cdot \lambda \in \mathbf{c}^s$ . Indeed, we can find  $z \cdot \lambda \in \mathbf{c}$  such that  $\mathbf{L}_{\lambda}^{\dot{z}}$  is a direct summand of  $\sigma$ , viewed as an object of  $\mathcal{C}^c \tilde{\mathcal{B}}^2$ ; then, by 5.5(a),  $\sigma$  is a summand of  $\overline{\mathcal{I}_s(\mathbf{L}_{\lambda}^{\dot{z}})}$ . If in addition,  $z \cdot \lambda \in \mathbf{c}^s$  then, by 5.6(a),(b), we have  $\overline{\mathcal{I}_s(\mathbf{L}_{\lambda}^{\dot{z}})} = \overline{\underline{\mathbf{h}}(\mathbb{L}_{\lambda, s}^{\dot{z}})}$  hence  $\sigma$  is a summand of  $\overline{\underline{\mathbf{h}}(\mathbb{L}_{\lambda, s}^{\dot{z}})}$  hence, by (a),  $\sigma$  is a summand of  $\overline{\tilde{\epsilon}_s \zeta(\underline{\chi}(\mathbb{L}_{\lambda, s}^{\dot{z}}))}$ , as required. If  $z \cdot \lambda \notin \mathbf{c}^s$  then, by 5.5(b), we have  $\mathcal{I}_s(\mathbf{L}_{\lambda}^{\dot{z}}) = 0$  which is a contradiction.) Since  $\mathbf{A} = \mathbf{A}'$ , from the uniqueness of decomposition of a semisimple algebra as a direct sum of simple algebras, we see that there is a unique bijection  $CS_{\mathbf{c}, s} \leftrightarrow \mathfrak{S}$ ,  $A \leftrightarrow \sigma_A$  such that  $\mathbf{A}_A = \mathbf{A}'_{\sigma_A}$  for any  $A \in CS_{\mathbf{c}, s}$ . From the definitions we now see that for any  $A \in CS_{\mathbf{c}, s}$  we have  $\overline{\tilde{\epsilon}_s \zeta(K)} \cong \sigma_A$ . Therefore, Theorem 7.3 holds.

**Theorem 7.4.** *We preserve the setup of Theorem 7.3. Let  $L \in \mathcal{C}^c Z_s$ ,  $K \in \mathcal{C}^c \tilde{G}_s$ . We have canonically*

$$(a) \quad \text{Hom}_{\mathcal{C}^c Z_s}(L, \underline{\zeta}(K)) = \text{Hom}_{\mathcal{C}^c \tilde{G}_s}(\underline{\chi}(L), K).$$

We can assume that  $L = \mathbb{L}_{\lambda,s}^{\dot{z}}$  where  $z \cdot \lambda \in \mathbf{c}^s$ . From 7.3 and its proof we see that

$$\mathrm{Hom}_{\mathcal{C}^c \tilde{G}_s}(\underline{\chi}(L), K) = \mathrm{Hom}_{\mathcal{Z}_{\mathbf{e}^s}}(\overline{\tilde{\epsilon}_s \zeta(\underline{\chi}(L))}, \overline{\tilde{\epsilon}_s \zeta(K)}) = \mathrm{Hom}_{\mathcal{Z}_{\mathbf{e}^s}}(\overline{\mathcal{I}_s(\mathbf{L}_{\lambda}^{\dot{z}})}, \overline{\tilde{\epsilon}_s \zeta(K)}).$$

Using 5.5(a) we see that

$$\mathrm{Hom}_{\mathcal{Z}_{\mathbf{e}^s}}(\overline{\mathcal{I}_s(\mathbf{L}_{\lambda}^{\dot{z}})}, \overline{\tilde{\epsilon}_s \zeta(K)}) \mathrm{Hom}_{\mathcal{C}^c \tilde{\mathcal{B}}^2}(\mathbf{L}_{\lambda}^{\dot{z}}, \tilde{\epsilon}_s \zeta(K)) = \mathrm{Hom}_{\mathcal{C}^c Z_s}(L, \underline{\zeta}(K)).$$

This proves the theorem.

**7.5.** We preserve the setup of Theorem 7.3. We show that for  $K \in \mathcal{C}^c \tilde{G}_s$  we have canonically

$$(a) \quad \mathfrak{D}(\underline{\zeta}(\mathfrak{D}(K))) = \underline{\zeta}(K).$$

Here the first  $\underline{\zeta}$  is relative to  $\tilde{\mathbf{c}}$ . It is enough to show that for any  $L \in \mathcal{C}^c Z_s$  we have canonically

$$\mathrm{Hom}_{\mathcal{C}^c Z_s}(L, \mathfrak{D}(\underline{\zeta}(\mathfrak{D}(K)))) = \mathrm{Hom}_{\mathcal{C}^c Z_s}(L, \underline{\zeta}(K)).$$

Here the left side equals

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^c \tilde{G}_s}(\underline{\zeta}(\mathfrak{D}(K)), \mathfrak{D}(L)) &= \mathrm{Hom}_{\mathcal{C}^c \tilde{G}_s}(\mathfrak{D}(K), \underline{\chi}(\mathfrak{D}(L))) \\ &= \mathrm{Hom}_{\mathcal{C}^c \tilde{G}_s}(\mathfrak{D}(K), \mathfrak{D}(\underline{\chi}(L))). \end{aligned}$$

(We have used 7.4(a) for  $\tilde{\mathbf{c}}$  and 6.7(b).) The right hand side equals

$$\mathrm{Hom}_{\mathcal{C}^c \tilde{G}_s}(\underline{\chi}(L), K) = \mathrm{Hom}_{\mathcal{C}^c \tilde{G}_s}(\mathfrak{D}(K), \mathfrak{D}(\underline{\chi}(L))).$$

(We have again used 7.4(a).) This proves (a).

**Theorem 7.6.** *Let  $s \in \mathbf{Z}_{\mathbf{c}}$ . Let  $K \in \mathcal{C}^c \tilde{G}_s$ . In  $\mathcal{C}^c \tilde{\mathcal{B}}^2$  we have*

$$\tilde{\epsilon}_s \underline{\zeta}(K) \cong \bigoplus_{z \cdot \lambda \in \mathbf{c}^s; z \cdot \lambda \underset{\text{left}}{\sim} \mathbf{e}^s(z^{-1}) \cdot \lambda} (\mathbf{L}_{\lambda}^{\dot{z}})^{\oplus N(z, \lambda)}$$

where  $N(z, \lambda) \in \mathbf{N}$ .

In  $\mathcal{C}^c Z_s$  we have

$$(a) \quad \underline{\zeta}(K) \cong \bigoplus_{z \cdot \lambda \in \mathbf{c}^s} (\mathbb{L}_{\lambda,s}^{\dot{z}})^{\oplus N(z, \lambda)}$$

where  $N(z, \lambda) \in \mathbf{N}$ . If  $N(z, \lambda) > 0$  then

$$\mathrm{Hom}_{\mathcal{C}^c Z_s}(\mathbb{L}_{\lambda,s}^{\dot{z}}, \underline{\zeta}(K)) \neq 0$$

hence by 7.4 we have  $\mathrm{Hom}_{\mathcal{C}^c \tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), K) \neq 0$  and in particular  $\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}) \neq 0$ . Using 6.5(d) we deduce that

$$(b) \quad z \cdot \lambda \underset{\text{left}}{\sim} \mathbf{e}^s(z^{-1}) \cdot \lambda.$$

Thus the direct sum in (a) can be restricted to  $z \cdot \lambda$  satisfying (b). We now apply  $\tilde{\epsilon}_s$  to both sides of (a) and use that  $\tilde{\epsilon}_s \mathbb{L}_{\lambda,s}^{\dot{z}} = \mathbf{L}_{\lambda}^{\dot{z}}$ . The theorem follows.

**7.7.** Let  $s \in \mathbf{Z}_{\mathbf{c}}$ . From 7.3 and 7.6 we see that any object of  $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$ , when viewed as an object of  $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ , is a direct sum of objects of the form  $\mathbf{L}_{\lambda}^z$  with  $z \cdot \lambda \in \mathbf{c}^s$  such that  $z \cdot \lambda \underset{\text{left}}{\sim} \mathbf{e}^s(z^{-1}) \cdot \lambda$ .

In the remainder of this subsection we assume that  $\tilde{G}$  is as in case A with  $G$  simple of type  $A_2$  (resp.  $B_2$  or  $G_2$ ). In this case  $W$  is generated by  $\sigma_1, \sigma_2$  in  $S$  with relation  $(\sigma_1\sigma_2)^m = 1$  where  $m = 3$  (resp.  $m = 4$  or  $m = 6$ ). We assume that  $\mathbf{c}$  is the two-sided cell of  $I$  consisting of all  $w \cdot 1$  where  $w \in W$ ,  $1 \leq |w| \leq m - 1$ . We shall write  $\mathbf{L}^{iji\dots}$  instead of  $\mathbf{L}_1^{\tilde{\sigma}_i\tilde{\sigma}_j\tilde{\sigma}_i\dots}$  where  $iji\dots$  is  $121\dots$  or  $212\dots$ . The objects of  $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$  of the form  $\tilde{\epsilon}_s\zeta(K)$  with  $K$  a simple object of  $\mathcal{C}^{\mathbf{c}}\tilde{G}_s$  are (up to isomorphism) the following ones:

$$\mathbf{L}^1 \oplus \mathbf{L}^2 \text{ for type } A_2;$$

$$\mathbf{L}^1 \oplus \mathbf{L}^2, \mathbf{L}^1 \oplus \mathbf{L}^{212}, \mathbf{L}^2 \oplus \mathbf{L}^{121}, \mathbf{L}^{121} \oplus \mathbf{L}^{212} \text{ for type } B_2;$$

$$\begin{aligned} &\mathbf{L}^1 \oplus \mathbf{L}^2, \mathbf{L}^1 \oplus \mathbf{L}^2 \oplus \mathbf{L}^{121} \oplus \mathbf{L}^{212}, \mathbf{L}^2 \oplus \mathbf{L}^{121} \oplus \mathbf{L}^{21212}, \\ &\mathbf{L}^1 \oplus \mathbf{L}^{212} \oplus \mathbf{L}^{12121}, \mathbf{L}^{121} \oplus \mathbf{L}^{212} \oplus \mathbf{L}^{12121} \oplus \mathbf{L}^{21212}, \mathbf{L}^{121} \oplus \mathbf{L}^{212} \text{ for type } G_2. \end{aligned}$$

Note that in type  $G_2$ ,  $\mathbf{L}^{121} \oplus \mathbf{L}^{212}$  comes from two nonisomorphic objects  $K$  of  $\mathcal{C}^{\mathbf{c}}\tilde{G}_s$ .

**7.8.** In this subsection we assume that  $\tilde{G}$  is as in case A with  $G = SL_2(\mathbf{k})$  and  $p \neq 2$ . In this case we may identify  $\mathbf{T} = \mathbf{k}^*$  and  $W = \{1, \sigma\}$  with  $\sigma(t) = t^{-1}$  for  $t \in \mathbf{T}$ . We take  $\tau \in \tilde{G}_1$  such that  $\mathbf{e} : G \rightarrow G$  in 2.3 satisfies  $\mathbf{e}(t) = t^q$  for any  $t \in T$ . Then for  $\lambda \in \mathfrak{s}_{\infty} \cong \mathbf{k}^*$  we have  $\mathbf{e}(\lambda) = \lambda^{q^{-1}}$ ,  $\sigma(\lambda) = \lambda^{-1}$ . Let  $\lambda_0$  be the unique element of  $\mathfrak{s}_{\infty}$  such that  $\lambda_0^2 = 1$ ,  $\lambda_0 \neq 1$ . In  $\mathbf{H}$  we have  $c_{1\cdot\lambda} = T_1 1_{\lambda}$  for all  $\lambda$ ,  $c_{\sigma\cdot\lambda} = v^{-1}T_{\sigma} 1_{\lambda}$  if  $\lambda \neq 1$ ,  $c_{\sigma\cdot 1} = v^{-1}T_{\sigma} 1_1 + v^{-1}T_1 1_1$ . It follows that the two-sided cells in  $I = \{w \cdot \lambda; w \in W, \lambda \in \mathfrak{s}_{\infty}\}$  are the following subsets of  $I$ :

$$\mathbf{c}_{\lambda} = \mathbf{c}_{\lambda^{-1}} = \{1 \cdot \lambda, 1 \cdot \lambda^{-1}, \sigma \cdot \lambda, \sigma \cdot \lambda^{-1}\} \text{ with } \lambda \in \mathfrak{s}_{\infty}; \lambda^2 \neq 1;$$

$$\mathbf{c}_{\lambda_0} = \{1 \cdot \lambda_0, \sigma \cdot \lambda_0\};$$

$$\mathbf{c}'_1 = \{\sigma \cdot 1\};$$

$$\mathbf{c}_1 = \{1 \cdot 1\}.$$

Let  $s \in \mathbf{Z}$ . The two-sided cells of  $I$  which are stable under  $\mathbf{e}^s$  are:

(i)  $\mathbf{c}_{\lambda} = \mathbf{c}_{\lambda^{-1}}$  where  $\lambda \in \mathfrak{s}_{\infty}$ ,  $\lambda^2 \neq 1$ ,  $\lambda^{q^{-s}} = \lambda$  (note that  $\mathbf{e}^s$  acts as 1 on this two-sided cell);

(ii)  $\mathbf{c}_{\lambda} = \mathbf{c}_{\lambda^{-1}}$  where  $\lambda \in \mathfrak{s}_{\infty}$ ,  $\lambda^2 \neq 1$ ,  $\lambda^{q^{-s}} = \lambda^{-1}$  (note that  $\mathbf{e}^s$  acts as a fixed point free involution on this two-sided cell and that we have necessarily  $s \neq 0$ );

(iii)  $\mathbf{c}_{\lambda_0}$  (note that  $\mathbf{e}^s$  acts as 1 on this two-sided cell);

(iv)  $\mathbf{c}'_1$  (note that  $\mathbf{e}^s$  acts as 1 on this two-sided cell);

(v)  $\mathbf{c}_1$  (note that  $\mathbf{e}^s$  acts as 1 on this two-sided cell).

For  $\mathbf{c}$  in (i)-(v), the  $\mathbf{e}^s$ -centre of  $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$  has exactly  $N$  simple objects (up to isomorphism) where  $N = 1$  in the cases (i),(ii),(iv),(v) and  $N = 4$  in the case (iii).



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